On the role of strongly connected components in argumentation

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Abstract

In argumentation theory, Dung's abstract framework provides a unifying view of several alternative semantics based on the notion of extension. Recently, a new semantics has been introduced to solve the problems related to counterintuitive results produced by literature proposals. In this semantics, extensions can be decomposed and constructed along the strongly connected components of the defeat graph. This paper proves that this property holds also in the context of all semantics encompassed by Dung's framework, showing that strongly connected components may play a general role in the definition and computation of argumentation semantics.

Keywords: Argumentation semantics, Extensions, Strongly Connected Components.

1 Introduction

Argumentation theory is a framework for practical and uncertain reasoning where commonsense reasoning, dealing with incomplete and uncertain information, is modeled as the process of constructing and comparing arguments for propositions. Since different arguments may support contradictory conclusions, the core problem consists in determining which arguments emerge undefeated from conflict, on the basis of a given *argumentation semantics*. In order to analyze and compare different kinds of semantics, Dung [4] has proposed an abstract framework able to encompass a large variety of proposals. Since all existing semantics give rise to counterintuitive results in some cases, in [1] a new semantics has been proposed, showing that it is able to overcome these limitations. The definition of this semantics exploits the notion of strongly connected components (SCCs) of the graph representing the argumentation framework, in order to constrain the set of extensions prescribed by the semantics. While this notion has not been previously considered in the context of other semantics, we show in this paper that in all semantics encompassed by Dung's framework there is a fundamental relationship between extensions and strongly connected components: all the definitions of extension given at a global graph level can be replaced by equivalent definitions given at the local level of strongly connected components.

The paper is organized as follows. Section 2 recalls the necessary basic concepts, Section 3, 4, 5, and 6 analyze the relationships between SCCs and Dung's admissible, complete, preferred, and grounded extensions respectively, while Section 7 concludes the paper.

2 Basic concepts

2.1 Argumentation framework

The general theory proposed by Dung [4] is based on the primitive notion of *argumentation framework*:

Definition 1 *An argumentation framework is a pair* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, where* $\mathcal{A}$ *is a set of arguments and* $\rightarrow$ *is a binary relation of 'attack' between them.*

In case $\mathcal{A}$ is finite, an argumentation framework $\langle \mathcal{A}, \rightarrow \rangle$ can be represented as a directed graph, called *defeat graph*, where nodes are the arguments and edges correspond to the elements of the attack relation $\rightarrow$.
In the following, nodes that attack a given $\alpha \in \mathcal{A}$ are called *defeaters* of α, and form a set denoted as parents(α):

Definition 2 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a node* $\alpha \in \mathcal{A}$*, we define* $\mathrm{parents}(\alpha) = \{\beta \in \mathcal{A} \mid \beta \rightarrow \alpha\}$*. If* $\mathrm{parents}(\alpha) = \emptyset$*, then* α *is called* initial.

Since we will frequently consider properties of sets of arguments, it is useful to extend to them the notations defined for the nodes. In particular, we extend the binary relation of attack and the notion of defeaters:

Definition 3 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, a node* $\alpha \in \mathcal{A}$ *and two sets* $S, P \subseteq \mathcal{A}$*, we define:*

$$
\begin{aligned}
&S \rightarrow \alpha \text{ iff } \exists \beta \in S : \beta \rightarrow \alpha \\
&\alpha \rightarrow S \text{ iff } \exists \beta \in S : \alpha \rightarrow \beta \\
&S \rightarrow P \text{ iff } \exists \alpha \in S, \beta \in P : \alpha \rightarrow \beta \\
&\mathrm{parents}(S) = \{\alpha \in \mathcal{A} \mid \alpha \rightarrow S\} \\
&\mathrm{parents}^*(S) = \{\alpha \in \mathcal{A} \mid \alpha \notin S \wedge \alpha \rightarrow S\}
\end{aligned}
$$

Given an argumentation framework, the core problem consists in computing the *defeat status* of its arguments, namely in determining which ones of them can be accepted in the framework, on the basis of a given *argumentation semantics*. Different argumentation semantics have been proposed in the literature, all of them relying on the notion of *extension*, which roughly consists in a set of arguments which do not conflict among them and which attack their attackers. Generally an argument is considered justified in a given semantics if it belongs to all extensions specified by the semantics itself. As pointed out in [7], the argumentation semantics can be distinguished in two classes: in the so-called unique-status approach there is exactly one extension for any argumentation framework, while in the multiple-status approach several extensions can generally be identified. The unique-status approach is adopted e.g. in the argumentation system introduced in [5], and is represented by the *grounded semantics* in Dung's framework, while the multiple-status approach is adopted e.g. in [6, 8, 2], and is captured by the *preferred semantics*.

2.2 Strongly connected components

While *strongly connected components* can be defined for generic directed (and undirected) graphs, we introduce them with reference to argumentation frameworks:

Definition 4 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, two nodes* $\alpha, \beta \in \mathcal{A}$ *are* path-equivalent *iff either* $\alpha = \beta$ *or there is a path from* α *to* β *and a path from* β *to* α*. The* strongly connected components *of* AF *are the equivalence classes of vertices under the relation of path-equivalence. The set of the strongly connected components of* AF *is denoted as* SCC(AF).

Given a node $\alpha \in \mathcal{A}$, we will indicate the SCC α belongs to as SCC(α). We extend to SCCs the notion of defeaters:

Definition 5 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and* $S \in \mathrm{SCC(AF)}$*,* $\mathrm{sccparents}(S) = \{P \in \mathrm{SCC(AF)} \mid P \neq S \wedge P \rightarrow S\}$*.*
If $\mathrm{sccparents}(S) = \emptyset$*,* S *is called* initial.

It is well-known that the graph obtained by considering SCCs as single nodes is acyclic. This property makes particularly useful the following definition.

Definition 6 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, a set* $E \subseteq \mathcal{A}$ *and* $S \in \mathrm{SCC(AF)}$*, we define:*

$$
\begin{aligned}
S^D(E) &= \{\alpha \in S \mid (E \cap \mathrm{parents}^*(S)) \rightarrow \alpha\} \\
S^P(E) &= \{\alpha \in S \mid (E \cap \mathrm{parents}^*(S)) \not\rightarrow \alpha \wedge \\
&\qquad \wedge \exists \beta \in (\mathrm{parents}^*(S) \cap \mathrm{parents}(\alpha)) : E \not\rightarrow \beta\} \\
S^U(E) &= S \setminus (S^D(E) \cup S^P(E))
\end{aligned}
$$

Notice that, according to this definition, $S^U(E) = \{\alpha \in S \mid (E \cap \text{parents}^*(S)) \not\rightarrow \alpha \wedge \forall \beta \in (\text{parents}^*(S) \cap \text{parents}(\alpha))\ E \rightarrow \beta\}$.

Taking into account that the SCCs of any argumentation framework make up an acyclic graph, it is easy to see that $S^D(E)$, $S^P(E)$ and $S^U(E)$ are determined only by the elements of E that belong to those SCCs that precede S in a topological sort. In particular, $S^D(E) = S^D(E \cap \text{parents}^*(S))$.

The sets defined above will play a key role in the analysis of the relationships between several traditional definitions of extensions and the decomposition of the defeat graph into its SCCs, carried out in the following sections.

3 Admissible sets

The following notions lie at the heart of argumentation framework theory [4].

Definition 7 *Given an argumentation framework* $\text{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *a set* $S \subseteq \mathcal{A}$ *is* conflict-free *iff* $\nexists \alpha, \beta \in S$ *such that* $\alpha \rightarrow \beta$.

Definition 8 *Given an argumentation framework* $\text{AF} = \langle \mathcal{A}, \rightarrow \rangle$*:*

- *An argument* $\alpha \in \mathcal{A}$ *is* acceptable *with respect to a set* $S \subseteq \mathcal{A}$ *iff* $\forall \beta \in \mathcal{A}$, *if* $\beta \rightarrow \alpha$ *then also* $S \rightarrow \beta$.
- *A set* $S \subseteq \mathcal{A}$ *is* admissible *iff* S *is conflict-free and each argument in* S *is acceptable with respect to* S, *i.e.* $\forall \beta \in \mathcal{A}$ *such that* $\beta \rightarrow S$ *we have that* $S \rightarrow \beta$.

In order to accomplish our analysis, it is necessary to extend the above definitions, referring them to specific subsets of $\mathcal{A}$.

Definition 9 *Let us consider an argumentation framework* $\text{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set of arguments* $A \subseteq \mathcal{A}$*:*

- *Given a set* $S \subseteq \mathcal{A}$, *an argument* $\alpha \in \mathcal{A}$ *is* acceptable against A *with respect to* S *iff* $\forall \beta \in A$, *if* $\beta \rightarrow \alpha$ *then also* $S \rightarrow \beta$.
- *A set* $S \subseteq \mathcal{A}$ *is* admissible against A *iff* S *is conflict-free and each argument in* S *is acceptable against* A *with respect to* S, *i.e.* $\forall \beta \in A : \beta \rightarrow S$ *we have that* $S \rightarrow \beta$.

Dung's fundamental lemma [4] shows that the admissibility property is preserved when including acceptable arguments. It is easy to prove that an analogous lemma holds with the generalized definitions.

Lemma 1 *Given an argumentation framework* $\text{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set* $A \subseteq \mathcal{A}$, *let* $S \subseteq \mathcal{A}$ *be a set of arguments admissible against* A, *and* $\alpha \in A$ *an argument acceptable against* A *with respect to* S. *If* $S \subseteq A$, *then* $S' = S \cup \{\alpha\}$ *is admissible against* A.

We now aim at showing that partitioning an admissible set along SCCs, one obtains subsets which are in turn admissible at the level of the SCCs themselves. On the other hand, composing sets which are admissible at the level of SCCs, one obtains an admissible set at the global level of the whole argumentation framework. This is achieved by Proposition 1, which requires two preliminary lemmas.

Lemma 2 *Given an argumentation framework* $\text{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *let* $E \subseteq \mathcal{A}$ *be an admissible set in* AF, *and* $\alpha \in \mathcal{A}$ *be an argument acceptable with respect to* E. *Denoting* $\text{SCC}(\alpha)$ *as* S, *we have that* $\alpha \in S^U(E)$ *and* α *is acceptable against* $(S^P(E) \cup S^U(E))$ *with respect to* $(E \cap S)$.

Proof. First of all, on the basis of Lemma 1 the set $(E \cup \{\alpha\})$ is admissible, and in particular conflict-free: as a consequence $\alpha \notin S^D(E)$, otherwise by the definition of $S^D(E)$ it would be the case that $E \rightarrow \alpha$. Moreover, $\alpha \notin S^P(E)$, otherwise by the definition of $S^P(E)$ we would have that $\exists \beta \in E : \beta \rightarrow \alpha$ and $E \not\rightarrow \beta$, thus contradicting the acceptability of α with respect to E. As a consequence, the only possibility for α is that $\alpha \in S^U(E)$.
Turning to the second part of the proof, since α is acceptable with respect to E we have that $\forall \beta \in (S^P(E) \cup S^U(E)) : \beta \rightarrow \alpha$ there is $\gamma \in E$ such that $\gamma \rightarrow \beta$: we have to prove that $\gamma \in S$ and therefore $\gamma \in (E \cap S)$. This is entailed by the fact that $\beta \in (S^P(E) \cup S^U(E))$, therefore, by definition of $S^P(E)$ and $S^U(E)$, all of its defeaters outside S does not belong to E. $\square$

Lemma 3 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *let* $E \subseteq \mathcal{A}$ *be a set of arguments such that,* $\forall S \in \mathrm{SCC}(\mathrm{AF})$: $(E \cap S) \subseteq S^U(E)$, *and* $(E \cap S)$ *is admissible against* $(S^P(E) \cup S^U(E))$.
If $\alpha \in S^U(E)$ *is an argument acceptable against* $(S^P(E) \cup S^U(E))$ *with respect to* $(E \cap S)$, *then* α *is acceptable with respect to* E.

Proof. We have to show that $\forall \beta \in \mathcal{A} : \beta \rightarrow \alpha$, $E \rightarrow \beta$. We distinguish two cases for β.
First, let us suppose that $\mathrm{SCC}(\beta) = \mathrm{SCC}(\alpha) \triangleq S$. If $\beta \in S^D(E)$, then $E \rightarrow \beta$ by definition of $S^D(E)$. If, on the other hand, $\beta \in (S^P(E) \cup S^U(E))$, then according to the hypothesis of acceptability concerning α it must be the case that $(E \cap S) \rightarrow \beta$.
Let us consider the other case, i.e. $\mathrm{SCC}(\beta) \neq \mathrm{SCC}(\alpha) \triangleq S$. In this case, $\beta \in (\mathrm{parents}^*(S) \cap \mathrm{parents}(\alpha))$, while by the hypothesis $\alpha \in S^U(E)$: on the basis of the definition of $S^U(E)$, it must be the case that $E \rightarrow \beta$. □

Proposition 1 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *let us consider a set of arguments* $E \subseteq \mathcal{A}$. *Then,* E *is admissible if and only if* $\forall S \in \mathrm{SCC}(\mathrm{AF})$ $(E \cap S) \subseteq S^U(E)$, *and* $(E \cap S)$ *is admissible against* $(S^P(E) \cup S^U(E))$.

Proof. First, let us prove that if E is admissible then it satisfies the conditions relevant to a generic $S \in \mathrm{SCC}(\mathrm{AF})$. According to the definition of admissible set, $\forall \alpha \in E$, α is acceptable with respect to E. As a consequence, on the basis of Lemma 2 we have in particular that $\forall \alpha \in (E \cap S)$, $\alpha \in S^U(E)$ and α is acceptable against $(S^P(E) \cup S^U(E))$ with respect to $(E \cap S)$. The first condition entails that $(E \cap S) \subseteq S^U(E)$. The second condition, as well as the fact that E is admissible and therefore conflict-free, entails that $(E \cap S)$ is admissible against $(S^P(E) \cup S^U(E))$.
As far as the other direction of the proof is concerned, let us first show that E is conflict-free by reasoning by contradiction, i.e. let us suppose that $\exists \alpha, \beta \in E : \beta \rightarrow \alpha$. Let us denote $\mathrm{SCC}(\alpha)$ as S. Clearly, it cannot be the case that $\mathrm{SCC}(\alpha) = \mathrm{SCC}(\beta)$, since in this case $(E \cap S)$ would not be conflict-free thus contradicting the hypothesis concerning its admissibility. As a consequence, $\beta \in (E \cap \mathrm{parents}^*(S))$, therefore $\alpha \in S^D(E)$ by the definition of $S^D(E)$. However, this contradicts the fact that $\alpha \in (E \cap S)$, which, by the hypothesis, is contained in $S^U(E)$.
In order to complete the proof, we have to prove that a generic $\alpha \in E$ is acceptable with respect to E. If we denote $\mathrm{SCC}(\alpha)$ as S, we have that $\alpha \in (E \cap S) \subseteq S^U(E)$, and since $(E \cap S)$ is admissible against $(S^P(E) \cup S^U(E))$, α is acceptable against $(S^P(E) \cup S^U(E))$ with respect to $(E \cap S)$. As a consequence, on the basis of Lemma 3 it must be the case that α is acceptable with respect to E. □

4 Complete semantics

The notion of complete extension is introduced in [4] as a unifying concept underlying various existing semantics. Due to space limitations, we directly introduce the definition in the context of the generalized framework.

Definition 10 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *let us consider two sets of arguments* $C, A \subseteq \mathcal{A}$. *A set* $S \subseteq \mathcal{A}$ *is a* complete extension in C against A *iff* $S \subseteq C$, S *is admissible against* A, *and every argument* $\alpha \in C$ *which is acceptable against* A *with respect to* S *belongs to* S. *The set of complete extensions in* C *against* A *will be denoted as* $\mathcal{CE}_{\mathrm{AF}}(C, A)$.

The original definition (Def. 23 of [4]) is recovered by letting $C = A = \mathcal{A}$, the set of complete extensions in this case will be denoted as $\mathcal{CE}_{\mathrm{AF}}$. The following proposition shows that also complete extensions are in correspondence with a decomposition along SCCs.

Proposition 2 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *let us consider a set of arguments* $E \subseteq \mathcal{A}$. *Then,* $E \in \mathcal{CE}_{\mathrm{AF}}$ *if and only if* $\forall S \in \mathrm{SCC}(\mathrm{AF})$, $(E \cap S) \in \mathcal{CE}_{\mathrm{AF}}(S^U(E), (S^P(E) \cup S^U(E)))$.

Proof. As for the first direction of the proof, if E is a complete extension then it is admissible, therefore by Proposition 1 we have that $\forall S \in \mathrm{SCC}(\mathrm{AF})$ $(E \cap S) \subseteq S^U(E)$, and $(E \cap S)$ is admissible against $(S^P(E) \cup S^U(E))$.

As a consequence, we have only to show that $\forall \alpha \in S^U(E)$ such that α is acceptable against $(S^P(E) \cup S^U(E))$ with respect to $(E \cap S)$, $\alpha \in (E \cap S)$. This follows from Lemma 3, which entails that α is acceptable with respect to E, and from the hypothesis that E is a complete extension, which entails that $\alpha \in E$ and therefore $\alpha \in (E \cap S)$.
As for the other direction of the proof, by Definition 10 we have that $\forall S \in \mathrm{SCC}(\mathrm{AF})$ $(E \cap S) \subseteq S^U(E)$, and $(E \cap S)$ is admissible against $(S^P(E) \cup S^U(E))$. Thus, Proposition 1 entails that E is admissible, therefore we have only to prove that $\forall \alpha \in \mathcal{A}$ such that α is acceptable with respect to E, $\alpha \in E$. Denoting $\mathrm{SCC}(\alpha)$ as S, by Lemma 2 we have $\alpha \in S^U(E)$, and α is acceptable against $(S^P(E) \cup S^U(E))$ with respect to $(E \cap S)$. Since by the hypothesis $(E \cap S) \in \mathcal{CE}_{\mathrm{AF}}(S^U(E), (S^P(E) \cup S^U(E)))$, it must be the case that $\alpha \in (E \cap S)$, and then that $\alpha \in E$. $\square$

5 Preferred semantics

Preferred semantics is introduced in [4] to overcome some limitations of stable semantics and is the most advanced proposal in the field of multiple-status approaches. It is based on the notion of preferred extension, defined as follows in the generalized framework.

Definition 11 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, let us consider two sets of arguments* $C, A \subseteq \mathcal{A}$*. A* preferred extension in C against A *is a maximal set* S *(with respect to set inclusion) such that* $S \subseteq C$ *and* S *is admissible against* A*. The set of preferred extensions in* C *against* A *will be denoted as* $\mathcal{FP}_{\mathrm{AF}}(C, A)$.

The original definition (Def. 7 of [4]) is recovered by letting $C = A = \mathcal{A}$, the set of preferred extensions in this case will be denoted as $\mathcal{FP}_{\mathrm{AF}}$. A relevant question concerns the existence of a preferred extension for any argumentation framework AF and for all sets $C, A \subseteq \mathcal{A}$. Since the empty set is admissible against any set, the positive answer is directly entailed by the following theorem, which generalizes a result in [4] (proof is omitted due to space limitations).

Theorem 1 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and two sets* $C, A \subseteq \mathcal{A}$*:*

- *The subsets of* C *that are admissible against* A *form a complete partial order.*
- *For all* $S \subseteq C$ *such that* S *is admissible against* A*, there is* $E \in \mathcal{FP}_{\mathrm{AF}}(C, A)$ *such that* $S \subseteq E$*.*

Also preferred extensions fit the decomposition schema along SCCs, as shown by Proposition 3, based on the following lemma.

Lemma 4 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, let* $E \subseteq \mathcal{A}$ *be an admissible set in* AF *and let* $S \in \mathrm{SCC}(\mathrm{AF})$*. Let* $\hat{E}$ *be a set of arguments such that* $(E \cap S) \subseteq \hat{E} \subseteq S^U(E)$*, and* $\hat{E}$ *is admissible against* $(S^U(E) \cup S^P(E))$*. Then, we have that* $(E \cup \hat{E})$ *is admissible in* AF*.*

Proof. First, we prove that $(E \cup \hat{E})$ is conflict-free. Since both E and $\hat{E}$ are conflict-free, we have to prove that $\hat{E} \not\rightarrow E$ and $E \not\rightarrow \hat{E}$. Since E is admissible, $\hat{E} \rightarrow E$ entails that $E \rightarrow \hat{E}$, therefore we have only to prove that $E \not\rightarrow \hat{E}$. Since $\hat{E} \subseteq S$, $\hat{E}$ can have attackers only in $\mathrm{parents}^*(S) \cup S$. Since $\hat{E} \subseteq S^U(E)$, $(E \cap \mathrm{parents}^*(S)) \not\rightarrow \hat{E}$, therefore $E \rightarrow \hat{E}$ only if $(E \cap S) \rightarrow \hat{E}$. However, this situation is not possible since $(E \cap S) \subseteq \hat{E}$ and $\hat{E}$ is conflict-free.
Now, we have to prove that $\forall \beta \in \mathcal{A}$ such that $\beta \rightarrow (E \cup \hat{E})$, it is the case that $(E \cup \hat{E}) \rightarrow \beta$. In case $\beta \rightarrow E$, the conclusion follows from admissibility of E. On the other hand, if $\beta \rightarrow \hat{E}$, we have that $\beta \in (\mathrm{parents}^*(S) \cup S)$ since $\hat{E} \subseteq S$; we distinguish three cases for β:

1. if $\beta \in \mathrm{parents}^*(S)$, then, taking into account that $\beta \rightarrow \hat{E}$ and $\hat{E} \subseteq S^U(E)$, it must be the case that $E \rightarrow \beta$;
2. if $\beta \in S^D(E)$, then according to the definition of $S^D(E)$ we have that $E \rightarrow \beta$;
3. if $\beta \in (S^P(E) \cup S^U(E))$, then since $\hat{E}$ is admissible against $(S^U(E) \cup S^P(E))$ we have that $\hat{E} \rightarrow \beta$.

In any case $(E \cup \hat{E}) \rightarrow \beta$, and we are done. $\square$

Proposition 3 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, let us consider a set of arguments* $E \subseteq \mathcal{A}$*. Then,* $E \in \mathcal{FP}_{\mathrm{AF}}$ *if and only if* $\forall S \in \mathrm{SCC(AF)}$, $(E \cap S) \in \mathcal{FP}_{\mathrm{AF}}(S^U(E), (S^P(E) \cup S^U(E)))$*.*

Proof. As far as the first direction of the proof is concerned, let us assume that $E \in \mathcal{FP}_{\mathrm{AF}}$. By definition, E is admissible, therefore, on the basis of Proposition 1, we have that $\forall S \in \mathrm{SCC(AF)}$, $(E \cap S) \subseteq S^U(E)$, and $(E \cap S)$ is admissible against $(S^P(E) \cup S^U(E))$. Let us reason by contradiction, assuming that $\exists \hat{S} \in \mathrm{SCC(AF)}$ such that $(E \cap \hat{S})$ is not maximal among the sets admissible against $(\hat{S}^P(E) \cup \hat{S}^U(E))$ that are included in $\hat{S}^U(E)$. According to Theorem 1, there must be a set $\hat{E}$ such that $(E \cap \hat{S}) \subset \hat{E} \subseteq \hat{S}^U(E)$, and $\hat{E}$ is admissible against $(\hat{S}^P(E) \cup \hat{S}^U(E))$. By Lemma 4, the set $E' \triangleq E \cup \hat{E}$ is admissible in AF, however it is easy to see that E is strictly contained in E', contradicting the maximality of E among the admissible sets of AF.
Let us turn now to the other direction of the proof, assuming that $\forall S \in \mathrm{SCC(AF)}$ $(E \cap S) \in \mathcal{FP}_{\mathrm{AF}}(S^U(E), (S^P(E) \cup S^U(E)))$. On the basis of Proposition 1, E is admissible: in order to prove that it is also a preferred extension, we reason again by contradiction, supposing that $\exists E' \subseteq \mathcal{A}, E \subset E' : E' \in \mathcal{FP}_{\mathrm{AF}}$ (notice that such a set exists by Theorem 1 restricted to the case $A = C = \mathcal{A}$). Since $E \subset E'$, there must be at least a $S \in \mathrm{SCC(AF)}$ such that $(E \cap S) \subset (E' \cap S)$: taking into account the acyclicity of the SCCs, there is in particular $\hat{S} \in \mathrm{SCC(AF)}$ such that

$$\forall S \in \mathrm{sccparents}(\hat{S})\ (E' \cap S) = (E \cap S) \quad (1)$$

$$(E \cap \hat{S}) \subset (E' \cap \hat{S}) \quad (2)$$

Since E' is admissible, Proposition 1 entails that $(E' \cap \hat{S}) \subseteq \hat{S}^U(E')$, and $(E' \cap \hat{S})$ is admissible against $(\hat{S}^P(E') \cup \hat{S}^U(E'))$. Taking into account (1), it is easy to see that $\hat{S}^U(E') = \hat{S}^U(E)$ and $\hat{S}^P(E') = \hat{S}^P(E)$, therefore $(E' \cap \hat{S}) \subseteq \hat{S}^U(E)$, and $(E' \cap \hat{S})$ is admissible against $(\hat{S}^P(E) \cup \hat{S}^U(E))$. However, on the basis of (2) we have that $(E \cap \hat{S}) \subset (E' \cap \hat{S})$, contradicting the hypothesis that $(E \cap \hat{S}) \in \mathcal{FP}_{\mathrm{AF}}(\hat{S}^U(E), (\hat{S}^P(E) \cup \hat{S}^U(E)))$.
□

6 Grounded semantics

The grounded semantics, originally introduced by Pollock [5], is probably the most representative proposal in the context of the unique-status approach. To relate grounded semantics with Dung's framework, a definition in terms of fixpoint of a so-called *characteristic function* is provided in [4]. We extend it to our generalized framework by introducing the notion of *characteristic function in a set C against a set A*:

Definition 12 *With reference to an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and two sets of arguments* $C, A \subseteq \mathcal{A}$*, the* characteristic function of AF in C against A*, denoted as* $\mathrm{F}_{\mathrm{AF},C,A}$*, is defined as follows:*

$$\mathrm{F}_{\mathrm{AF},C,A} : 2^C \rightarrow 2^C$$
$$\mathrm{F}_{\mathrm{AF},C,A}(S) = \{\alpha \mid \alpha \in C,$$
$$\alpha \textit{ acceptable against } A \textit{ with respect to } S\}$$

It is easy to see that $\mathrm{F}_{\mathrm{AF},C,A}$ is monotonic (with respect to set inclusion). The grounded extension is then the least fixed point of this function.

Definition 13 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and two sets* $C, A \subseteq \mathcal{A}$*, the* grounded extension of AF in C against A*, denoted as* $\mathrm{GE}_{\mathrm{AF}}(C, A)$*, is the least fixed point (with respect to set inclusion) of* $\mathrm{F}_{\mathrm{AF},C,A}$.

Notice that, by definition, $\mathrm{GE}_{\mathrm{AF}}(C, A) \subseteq C$. The original definition (Def. 20 of [4]) is recovered by letting $C = A = \mathcal{A}$, the grounded extension in this case will be denoted as $\mathrm{GE}_{\mathrm{AF}}$.

Also the existence of the grounded extension is guaranteed (proof is omitted).

Lemma 5 *For any argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and for all sets* $C, A \subseteq \mathcal{A}$*,* $\mathrm{GE}_{\mathrm{AF}}(C, A)$ *exists and is unique.*

The following result, concerning the relationship between grounded and complete extensions, will be exploited to complete our analysis (proof is omitted).

Proposition 4 *Let us consider an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, and two sets of arguments* $C, A \subseteq \mathcal{A}$*. If* $C \subseteq A$*, then* $\mathrm{GE}_{\mathrm{AF}}(C, A)$ *is the least (with respect to set inclusion) complete extension in* C *against* A*.*

We are now in a position to apply our decomposition scheme also to grounded semantics.

Proposition 5 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, let us consider a set of arguments* $E \subseteq \mathcal{A}$*. Then,* $E = \mathrm{GE}_{\mathrm{AF}}$ *if and only if* $\forall S \in \mathrm{SCC}(\mathrm{AF})$ $(E \cap S) = \mathrm{GE}_{\mathrm{AF}}(S^U(E), (S^P(E) \cup S^U(E)))$.

Proof. As to the first part of the proof, suppose that $E = \mathrm{GE}_{\mathrm{AF}}$. By Proposition 4, E is a complete extension, thus Proposition 2 entails that $\forall S \in \mathrm{SCC}(\mathrm{AF})$, $(E \cap S) \in \mathcal{CE}_{\mathrm{AF}}(S^U(E), (S^P(E) \cup S^U(E)))$. Taking into account Proposition 4, we have to prove that $\forall S \in \mathrm{SCC}(\mathrm{AF})$ $(E \cap S)$ is the least complete extension in $S^U(E)$ against $(S^P(E) \cup S^U(E))$. We reason by contradiction, supposing that there is at least one SCC where the thesis is not verified. In particular, since the SCCs of AF make up an acyclic graph, we can choose $\hat{S} \in \mathrm{SCC}(\mathrm{AF})$ such that:
i) $\forall S \in \mathrm{SCC}(\mathrm{AF})$ such that S is antecedent to $\hat{S}$ in the graph of the SCCs, $(E \cap S) = \mathrm{GE}_{\mathrm{AF}}(S^U(E), (S^P(E) \cup S^U(E)))$;
ii) $\exists \hat{E} \subset (E \cap \hat{S})$, $\hat{E} = \mathrm{GE}_{\mathrm{AF}}(\hat{S}^U(E), (\hat{S}^P(E) \cup \hat{S}^U(E)))$.
Note that in case $\hat{S}$ is initial, the first condition is trivially verified. Moreover, the second condition follows from the fact that, on the basis of Lemma 5, the grounded extension in $\hat{S}^U(E)$ against $(\hat{S}^P(E) \cup \hat{S}^U(E))$ must exist, and according to Proposition 4 it is included in all the elements of $\mathcal{CE}_{\mathrm{AF}}(S^U(E), (S^P(E) \cup S^U(E)))$.
Now, taking again into account that the SCCs of AF make up an acyclic graph, it is easy to see that it is possible to construct a set E' such that:
i) $\forall S \in \mathrm{SCC}(\mathrm{AF})$ such that S is antecedent to $\hat{S}$ in the graph of the SCCs, $(E' \cap S) = (E \cap S)$;
ii) $(E' \cap \hat{S}) = \hat{E}$;
iii) $\forall S \in \mathrm{SCC}(\mathrm{AF})$ $(E' \cap S) = \mathrm{GE}_{\mathrm{AF}}(S^U(E'), (S^P(E') \cup S^U(E')))$.
In fact, it is possible to construct a set E'_* satisfying the first two conditions. Thus, for any SCC S which either precedes $\hat{S}$ or is equal to $\hat{S}$, it turns out that $S^U(E'_*) = S^U(E)$ and $S^P(E'_*) = S^P(E)$: as a consequence, taking into account the properties of E and $\hat{E}$ stated above, E'_* satisfies the third condition relevant to all such SCCs. Now, E' can be obtained constructively starting from E'_* by proceeding along the other SCCs of the defeat graph, taking into account that $\forall S \in \mathrm{SCC}(\mathrm{AF})$ $\mathrm{GE}_{\mathrm{AF}}(S^U(E'), (S^P(E') \cup S^U(E')))$ always exists by Lemma 5.
Now, by Proposition 4, $\forall S \in \mathrm{SCC}(\mathrm{AF})$ $(E' \cap S) \in \mathcal{CE}_{\mathrm{AF}}(S^U(E'), (S^P(E') \cup S^U(E')))$. As a consequence, on the basis of Proposition 2, E' is a complete extension, while since $(E' \cap \hat{S}) = \hat{E} \subset (E \cap \hat{S})$ it is not true that $E \subseteq E'$. However, this contradicts the hypothesis that E is the grounded extension of AF, and as such the least complete extension of AF (see Proposition 4).
As to the other direction of the proof, suppose that $\forall S \in \mathrm{SCC}(\mathrm{AF})$ $(E \cap S) = \mathrm{GE}_{\mathrm{AF}}(S^U(E), (S^P(E) \cup S^U(E)))$.
By Proposition 4 and Proposition 2, E is a complete extension of AF, therefore we have only to prove that it is the least complete extension (see Proposition 4). We reason by contradiction, assuming that the grounded extension E', which is a complete extension by Proposition 4, is strictly included in E. Thus, there must be at least a SCC S such that $(E' \cap S) \subset (E \cap S)$: since the SCCs form an acyclic graph, there is a SCC $\hat{S}$ such that:

$$\forall S \in \mathrm{sccparents}(\hat{S}) \ (E' \cap S) = (E \cap S) \quad (3)$$

$$(E' \cap \hat{S}) \subset (E \cap \hat{S}) \quad (4)$$

Moreover, on the basis of Proposition 2 applied to $\hat{S}$ it must be the case that $(E' \cap \hat{S}) \in \mathcal{CE}_{\mathrm{AF}}(\hat{S}^U(E'), (\hat{S}^P(E') \cup \hat{S}^U(E')))$. Taking into account (3), it is easy to see that $\hat{S}^U(E') = \hat{S}^U(E)$ and $\hat{S}^P(E') = \hat{S}^P(E)$, therefore $(E' \cap \hat{S}) \in \mathcal{CE}_{\mathrm{AF}}(\hat{S}^U(E), (\hat{S}^P(E) \cup \hat{S}^U(E)))$. However, according to (4) we have that $(E' \cap \hat{S})$ is strictly included in $(E \cap \hat{S})$, contradicting the hypothesis that $(E \cap \hat{S}) = \mathrm{GE}_{\mathrm{AF}}(\hat{S}^U(E), (\hat{S}^P(E) \cup \hat{S}^U(E)))$ and therefore, on the basis of Proposition 4, that $(E \cap \hat{S})$ is the least complete extension in $\hat{S}^U(E)$ against $(\hat{S}^P(E) \cup \hat{S}^U(E))$. $\square$

7 Conclusions

We have shown that the notion of extension introduced in various argumentation approaches, such as [4, 5, 6, 7], can be equivalently defined in terms of the union of sets corresponding to a specialized notion of the same extension at the level of strongly connected components. As to our knowledge, the use of SCCs to define the extensions has only been considered in [3], where, however, the direction of graph edges is not taken into account, thus limiting the results to the case of totally unrelated subgraphs. In [2] recursion is applied at the level of the argument structure rather than at the level of the defeat graph; clearly, these levels are completely independent. Our result can be useful for the development of efficient algorithms based on a principle of local computation. In particular, known algorithms for the computation of preferred extensions [3] are based on backtracking at the graph level, therefore it can be reasonably expected that a significant reduction of the search space can be achieved. Similarly, this result can be useful to the study of incremental algorithms, in the usual case where the defeat graph dynamically evolves. More importantly, it is particularly significant that the semantics introduced in [1] has been directly defined in terms of SCCs. In particular, it simply relies on the notion of maximal conflict free sets, at the level of SCCs, to constrain the prescribed extensions:

Definition 14 *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, a set* $E \subseteq \mathcal{A}$ *is an extension, denoted as* $E \in \mathcal{FM}(\mathrm{AF})$*, iff* $\forall S \in \mathrm{SCC}(\mathrm{AF})$

$$S \cap E \begin{cases} \in \mathcal{FI}(\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))}) \\ \quad \textit{if } |\mathrm{SCC}(\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))})| = 1 \\ \in \mathcal{FM}(\mathrm{AF}{\downarrow}_{(S^P(E) \cup S^U(E))}) \\ \quad \textit{otherwise} \end{cases}$$

where $\mathcal{FI}(\mathrm{AF})$ denotes the set of maximal conflict-free sets of AF, and $\mathrm{AF}{\downarrow}_S$ denotes the restriction of AF to its subset S. Due to space limitations, the reader is referred to [1] for a detailed discussion of this novel semantics, which exhibits the desired behavior in several problematic examples where other semantics provide counterintuitive results. In particular, while maintaining the same capability of discriminating floating arguments as preferred semantics, it correctly deals with odd-length cycles as grounded semantics: it builds on intuitions coming from both grounded and preferred semantics, combining the advantages of both of them. As a consequence, it is reasonable to conjecture that further investigations on argumentation semantics centered on the notion of strongly connected components will give significant results and insights concerning foundations of argumentation theory.

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