Extending consonant approximations to capacities

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Abstract

Results concerning consonant approximations are available in the literature only for the case of belief functions. This paper investigates the extension of consonant approximations to the case of capacities, which feature less specific properties and can be considered as the most general family of set measures suitable to uncertainty representation. It is shown that existing results for belief functions can be properly extended. In the meanwhile, some imprecisions in the previous literature are pointed out and corrected.

Keywords: Consonant approximations, capacities, belief functions.

1 Introduction

The issue of computing an approximation of an uncertainty assessment, expressed in a given formalism, in terms of a simpler formalism attracts the interest of researchers for several reasons. On the one hand, transforming a more complex representation into a simpler one gives rise to computational advantages. On the other hand, transformation procedures are a key issue for enabling interoperability among heterogeneous uncertain reasoning systems, as is the case in *multi-agent systems* [1]. Possibility theory [6] is among the most reasonable choices as a formalism for computing approximations of other representations. In fact, a possibility measure can be expressed in terms of n parameters, while more complex formalisms require 2^n. Moreover, the properties of this formalism have been deeply investigated and a number of applications are reported in the literature. The problem of approximating a belief function with a possibility (called *consonant approximation*) has been thoroughly analyzed in [7], and considered subsequently in [8, 9]. Alternative approaches to approximation of belief functions include reducing the number of focal elements [3], and coarsening the frame of discernment [4]. As to our knowledge, no results are available in the literature about consonant approximations of more general uncertainty representations. Since capacities can be understood as a very general family of set measures suitable to uncertainty representation, this paper aims at investigating the issue of consonant approximations of capacities. The contribution provided consists in the generalization of the main results previously limited to belief functions. While carrying out the study, some little imperfections in the results presented in [7] were identified. Therefore, besides generalizing the treatment of consonant approximations, this paper provides also some adjustments to the existing literature. The paper is organized as follows. After recalling some basic concepts in section 2, in section 3 previous literature proposals concerning consonant approximations are reviewed. Section 4 discusses families of consonant approximations of capacities, while in section 5 minimal consonant approximations of capacities are characterized. Finally, section 6 summarizes the paper.

2 Basic concepts

The symbol Ω denotes a *finite* set, called *universal set*, of pairwise disjoint (non-impossible) elementary events whose union is the certain event: $\Omega = \{\omega_1, \ldots, \omega_N\}$. Then $\omega_1, \ldots, \omega_N$ are called *atoms*, 2^Ω is the powerset of Ω, $|A|$ is the cardinality of $A \in 2^\Omega$, $\overline{A} = (\Omega \setminus A)$ denotes the complement of A. Though more general definitions are possible, in this paper an uncertainty assessment is defined as a function $f : 2^\Omega \to [0,1]$.
An uncertainty assessment should at least satisfy a basic monotonicity requirement and two boundary conditions: this gives rise to the notion of (1-monotone) capacity.

A function $C : 2^\Omega \to [0,1]$ is a (normalized) *capacity* [2] whenever: $C(\emptyset) = 0$; $C(\Omega) = 1$; $C(A) \le C(B), \forall A, B \in 2^\Omega$ such that $A \subset B$.

Belief functions [10], and in particular *necessity measures* [6] are special cases of capacities. Their definitions are well known; we shall recall in Proposition 1 their characterizations in terms of their Möbius inverses [2].
For any $f : 2^\Omega \to \mathbb{R}$, there is a one-to-one correspondence between f and its *Möbius inverse* or *mass function* $m : 2^\Omega \to \mathbb{R}$, according to the following equations [2]:
$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} f(B)$,
$f(A) = \sum_{B \subseteq A} m(B)$.
The events $A \in 2^\Omega$ such that $m(A) \neq 0$ are called *focal elements* (or *focal sets*). The set of focal elements will be denoted as $\mathcal{F}$.

Proposition 1 *Given $f : 2^\Omega \to \mathbb{R}$, let m be its Möbius inverse. Then*
(a) f is a capacity *iff m is such that:*
$m(\emptyset) = 0$; $\sum_{B \in 2^\Omega} m(B) = 1$;
$\sum_{\omega \in B \subseteq A} m(B) \ge 0$, $\forall A \in 2^\Omega, \forall \omega \in A$.
Further, if f is a capacity and $\mathcal{F}$ the set of its focal elements, then
(b) f is a belief function *iff m is non-negative;*
(c) f is a necessity measure *iff $\mathcal{F}$ is totally ordered by relation '$\subset$' (this property is called* consonance*: necessity measures are also referred to as* consonant belief functions*).*

The *conjugate* C' of a capacity C is defined by $C'(A) = 1 - C(\overline{A})$, $\forall A \in 2^\Omega$. The conjugate of a belief function is called a *plausibility function*, the conjugate of a necessity measure is called a *possibility measure*. As usual, a belief function will be denoted as Bel, a plausibility function as Pl, a necessity measure as N, and a possibility measure as Π. A possibility measure Π is completely specified by a possibility distribution defined over the atoms $\pi : \Omega \to [0,1]$, since it holds that
$\Pi(A) = \max_{\omega \in A}\{\pi(\omega)\}$, $\forall A \in 2^\Omega$.
The normalization condition then implies: $\exists \omega \in \Omega : \pi(\omega) = 1$.

3 Previous proposals

First, let us recall the notion of (weak) inclusion [7]: a belief function Bel is included in a belief function Bel' iff $Bel(A) \ge Bel'(A)$, $\forall A \in 2^\Omega$. This is equivalent to the condition $Pl(A) \le Pl'(A)$, $\forall A \in 2^\Omega$, and therefore to the inclusion of the interval $[Bel(A), Pl(A)]$ within $[Bel'(A), Pl'(A)]$. Clearly this definition is applicable to any uncertainty representation which encompasses a lower and an upper measure, related by conjugacy.
Both inner and outer consonant approximations of belief functions are considered in [7]. An inner consonant approximation of a belief function Bel is a necessity measure N such that $N(A) \ge Bel(A)$, (equivalently, $\Pi(A) \le Pl(A)$), $\forall A \in 2^\Omega$. Such an approximation exists [7] only if the intersection of all the focal sets of Bel is not empty: $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$. In this case, $\max_{\omega \in \Omega} Pl(\omega) = 1$ and the optimal inner approximation is obtained from the following possibility distribution: $\pi(\omega) = Pl(\omega)$, $\forall \omega \in \Omega$ [5]. In the cases where this approximation is not directly applicable, a simple approach consists in properly modifying the belief function and then applying the above procedure. To this purpose, in order to obtain a suitable mass assignment m' from the original m, it is suggested in [8, 9] to select an atom ω_i and let $m'(A \cup \{\omega_i\}) = m(A)$, $\forall A \in \mathcal{F}$. As a consequence, $\bigcap_{F' \in \mathcal{F}'} F' = \{\omega_i\}$. Thus, the value of $Pl(\omega_i)$ is raised to 1, while the plausibility values of other atoms are unchanged. As a consequence, $\Pi'(A) = 1$, $\forall A : \omega_i \in A$, $\Pi'(A) = \max_{\omega \in A} Pl(\omega)$, $\forall A : \omega_i \notin A$. A natural choice is selecting ω_i such that
$Pl(\omega_i) \ge Pl(\omega)$, $\forall \omega \in \Omega$.

An outer consonant approximation of a belief function Bel is a necessity measure N such that $N(A) \leq Bel(A)$, (equivalently, $\Pi(A) \geq Pl(A)$), $\forall A \in 2^{\Omega}$. In section 3.3 of [7] the following procedure is introduced to derive a family of necessity measures including a given belief function. Let $\mathcal{F} = \{F_1, \ldots, F_{|\mathcal{F}|}\}$ be the set of focal elements of a belief function Bel defined on $\Omega = \{\omega_1, \ldots, \omega_n\}$. Let σ be a permutation of $\{1, \ldots, n\}$ and define, for $1 \leq j \leq n$, $E_j^{\sigma} = \{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(j)}\}$, i.e. the set composed by the first j atoms in σ, and $\mathcal{E}^{\sigma} = \{E_1^{\sigma}, \ldots, E_n^{\sigma}\}$, i.e. the set composed by all the n sets E_j^{σ}. A consonant mass assignment m_{σ} is obtained from σ through the following procedure: for any focal set F_i of Bel, let $f_{\sigma}(i) = \min\{j : F_i \subseteq E_j^{\sigma}\}$, and then define

$$m_{\sigma}(E_j^{\sigma}) = \sum_{i:f_{\sigma}(i)=j} m(F_i) \qquad (1)$$

We will denote as N_{σ} the necessity measure whose Möbius inverse is the mass function m_{σ} derived from a permutation σ by (1), and as Π_{σ} the possibility measure conjugate of N_{σ}. As shown in [7], for any σ the necessity measure N_{σ} includes the belief function Bel (i.e. $N_{\sigma}(A) \leq Bel(A)$, $\forall A \in 2^{\Omega}$).

With the aim of identifying outer approximations which are minimal with respect to inclusion, in [7] attention is then focused on an analogous procedure concerning permutations of focal sets. Let ρ be a permutation of the focal sets of Bel, and define correspondingly the following family of sets: $S_1^{\rho} = F_{\rho(1)}$, $S_2^{\rho} = F_{\rho(1)} \cup F_{\rho(2)}$, ..., $S_{|\mathcal{F}|}^{\rho} = F_{\rho(1)} \cup \ldots \cup F_{\rho(|\mathcal{F}|)}$. We define also $S_0^{\rho} = \emptyset$. Then a mass assignment m_{ρ} is obtained in a way analogous to (1): let $f_{\rho}(i) = \min\{j : F_i \subseteq S_j^{\rho}\}$, and then define

$$m_{\rho}(S_j^{\rho}) = \sum_{i:f_{\rho}(i)=j} m(F_i). \qquad (2)$$

We will denote as N_{ρ} the necessity measure whose Möbius inverse is the mass function m_{ρ} derived from a permutation ρ by (2), and as Π_{ρ} the possibility measure conjugate of N_{ρ}.

In [7] the relationships between the approximations generated by these two families of permutations are then analyzed and, in particular, results concerning minimal (with respect to inclusion) outer consonant approximations are obtained. In the following two sections we investigate respectively the extension of the families of consonant approximations to the case of capacities and the characterization of minimal outer consonant approximations in this generalized context.

4 Consonant approximations of capacities

First of all, some precisations are needed when considering the notion of consonant approximation of a capacity. Given a capacity C and its conjugate C', let us say that C is *upper conjugate* iff $C(A) \geq C'(A)$, $\forall A \in 2^{\Omega}$, i.e. C dominates its conjugate. Then, C' is called *lower conjugate*. In this case, the notions of inner/outer consonant approximation based on the inclusion relationships between intervals $[N(A), \Pi(A)]$ and $[Bel(A), Pl(A)]$, can be directly extended referring to the interval $[C'(A), C(A)]$. However, it may be the case that a capacity neither dominates nor is dominated by its conjugate. In this case, the notion has to be weakened: for an inner approximation we only require that $\Pi(A) \leq C(A)$, $\forall A \in 2^{\Omega}$, while for an outer approximation it must hold that $N(A) \leq C(A)$. In the following, we will refer to these weakened requirements which are more general. It is easy to see that the weaker notions imply anyway interval inclusion when the considered capacity is upper or lower conjugate respectively.

Extending inner consonant approximations to capacities is straightforward. In fact, the inner approximation proposed in [7] can be applied to any capacity such that $\max_{\omega \in \Omega} C(\{\omega\}) = 1$. The resulting possibility measure Π is still an inner approximation, since $C(A) \geq \max_{\omega \in A} C(\{\omega\}) = \Pi(A)$. Moreover, for any possibility measure Π' such that $\Pi'(A) \leq C(A)$, $\forall A \in 2^{\Omega}$ we have that $\Pi'(\{\omega\}) \leq C(\{\omega\}) = \Pi(\{\omega\})$, $\forall \omega \in \Omega$ and consequently $\Pi'(A) \leq \Pi(A)$, $\forall A \in 2^{\Omega}$. Therefore this inner approximation is optimal (maximal with respect to inclusion) for any capacity. When the above applicability condition is not satisfied, the solution[8, 9] of selecting $\omega_i : C(\{\omega_i\}) \geq C(\{\omega\})$, $\forall \omega \in \Omega$, putting $\pi(\omega_i) = 1$ and $\pi(\omega) = C(\{\omega\})$ $\forall \omega \in \Omega, \omega \neq \omega_i$, can still be applied. It has however to be noted that the resulting possibility is neither an inner nor an outer approximation with respect to the original assignment.

Turning to outer approximations, first of all, let us show that permutations of atoms still give rise to a family of outer consonant approximations when a generic capacity C is considered instead of a belief function.

Proposition 2 *For any capacity C defined on 2^Ω, for any permutation σ of atoms, the necessity measure N_σ whose Möbius inverse is the mass function m_σ defined by (1) is such that $N_\sigma(A) \leq C(A)$, $\forall A \in 2^\Omega$.*

First of all, it has to be noted that m_σ is non-negative also in this case: in fact, by construction $m_\sigma(E_j^\sigma) = \sum_{\omega_{\sigma(j)} \in A_i \subseteq E_j^\sigma} m(A_i)$, and this sum is non-negative for any capacity, by property 1(a). Then we have that:

$$N_\sigma(A) = \sum_{E_j^\sigma \in \mathcal{E}^\sigma, E_j^\sigma \subseteq A} m_\sigma(E_j^\sigma) =$$

$$= \sum_{E_j^\sigma \in \mathcal{E}^\sigma, E_j^\sigma \subseteq A} \left(\sum_{F_i \in \mathcal{F}, f_\sigma(i) = j} m(F_i) \right). \qquad (3)$$

Now, by construction of the sets E_j^σ (and letting $E_0^\sigma = \emptyset$) there is a value $0 \leq q \leq \mid A \mid$ such that $\forall j \leq q$, $E_j^\sigma \subseteq A$, while $\forall j > q, E_j^\sigma \not\subseteq A$. Therefore the sum in (3) reduces to

$$N_\sigma(A) = \sum_{F_i \in \mathcal{F}, F_i \subseteq E_q^\sigma} m(F_i) = C(E_q^\sigma) \leq C(A). \square$$

As a consequence, the following proposition (extending Proposition 2 of [7]) holds.

Proposition 3 *Given a capacity C, for any necessity measure N such that $C(A) \geq N(A)$, $\forall A \in 2^\Omega$, there is a permutation σ such that $N_\sigma(A) \geq N(A)$, $\forall A \in 2^\Omega$.*

We omit the proof, which is a direct extension of the one of [7], page 428, where the statement refers to a belief function, but the demonstration does not rely on any of its specific properties and is valid for any capacity.

Let us now show that also the procedure based on permutations of focal sets preserves its desirable properties in the context of capacities.

Proposition 4 *For any capacity C defined on 2^Ω, for any permutation ρ of its focal elements: (a) m_ρ is non-negative;*
(b) $N_\rho(A) \leq C(A)$, $\forall A \in 2^\Omega$.

As for part (a), we have that for $1 \leq j \leq |\mathcal{F}|$:

$$m_\rho(S_j^\rho) = \sum_{F_i \subseteq S_j^\rho, F_i \not\subseteq S_{j-1}^\rho} m(F_i) =$$

$$= \sum_{F_i \subseteq S_j^\rho} m(F_i) - \sum_{F_i \subseteq S_{j-1}^\rho} m(F_i) =$$

$$= C(S_j^\rho) - C(S_{j-1}^\rho) \geq 0.$$

As for part (b), let $0 \leq k \leq |\mathcal{F}|$ be the maximum index such that $S_k^\rho \subseteq A$, then

$$N_\rho(A) = \sum_{j \leq k} m_\rho(S_j^\rho) =$$

$$= \sum_{F \subseteq S_k^\rho} m(F) = C(S_k^\rho) \leq C(A). \square$$

To characterize the relationships between the two families of consonant approximations, Proposition 3 of [7] states that
$\forall \rho, \exists \sigma : m_\rho(A) = m_\sigma(A)$, $\forall A \in \Omega$.
However, the proof is flawed and actually neither this relationship (nor the converse) hold. In fact, consider a case where $\Omega \in \mathcal{F}$: any permutation of focal elements ρ which selects Ω as first element, gives rise to the maximally vague assignment $m_\rho(\Omega) = 1$, $m_\rho(A) = 0$ for any other $A \in 2^\Omega$. Then, it is sufficient that $\mathcal{F}$ includes at least two other distinct focal elements F_1 and F_2 to prevent that there is a permutation of atoms σ, such that $m_\rho = m_\sigma$. In fact, for any σ there is a set $E_j^\sigma \neq \Omega$ such that $f_\sigma(1) = j \vee f_\sigma(2) = j$, as a consequence $m_\sigma(E_j^\sigma) > 0$, $m_\sigma(\Omega) < 1$.
On the other hand, consider any mass assignment such that for two atoms ω_1, ω_2 it holds that $m(\{\omega_1\}) = 0$, $m(\{\omega_2\}) > 0$, $m(\{\omega_1, \omega_2\}) = 0$ and consider a permutation σ such that ω_1 and ω_2 are at the first and second place respectively. We have then $E_1^\sigma =$

$\{\omega_1\}$ with $m_\sigma(E_1^\sigma) = 0$ and $E_2^\sigma = \{\omega_1, \omega_2\}$ with $m_\sigma(E_2^\sigma) = m(\{\omega_2\}) > 0$. However, there can not be any ρ such that $S_j^\rho = \{\omega_1, \omega_2\}$ for any j, therefore, $m_\rho(\{\omega_1, \omega_2\}) = 0$, for any ρ. This imperfection gives rise to the need of a reformulation of the characterization of minimal (with respect to inclusion) consonant approximations as proposed in [7], which is carried out in the next section.

5 Minimal approximations

First of all, let us identify those approximations N_ρ which are not minimal. This is dealt with by Proposition 4 of [7], whose formulation is however slightly imprecise: the following proposition extends the result to capacities also correcting its statement.

Proposition 5 *Let ρ a permutation of the focal sets and Π_ρ the relevant possibility measure defined according to (2). If the permutation ρ is such that $F_{\rho(i)} \subseteq S_k^\rho$ for some $k < i$, $F_{\rho(i)} \not\subseteq S_{k-1}^\rho$ and $(F_{\rho(i)} \setminus S_{k-1}^\rho) \subsetneq (S_k^\rho \setminus S_{k-1}^\rho)$, then there is another permutation τ such that $\Pi_\tau(\{\omega\}) \leq \Pi_\rho(\{\omega\})$, $\forall \omega \in \Omega$ (and hence $\Pi_\tau(A) \leq \Pi_\rho(A)$, $\forall A \in 2^\Omega$).*

The permutation τ is obtained from ρ exchanging positions i and k as follows: $\tau(j) = \rho(j)$, for $j < k$ and $j > i$; $\tau(k) = \rho(i)$; $\tau(j) = \rho(j-1)$ for $k < j \leq i$. It follows that $S_j^\tau = S_j^\rho$ for $j < k$ and $j > i$; $S_k^\tau \subsetneq S_k^\rho$; and $S_j^\tau = S_{j-1}^\rho$ for $k < j \leq i$ (noting in particular that $S_i^\tau = S_i^\rho = S_{i-1}^\rho$). Since, by (2), the mass allocated to a given set in the sequence depends only on the set itself and on the previous one in the sequence, we have in turn that $m_\rho(S_j^\tau) = m_\tau(S_j^\rho)$ for $j < k$ and $j > i$; $m_\tau(S_k^\tau) + m_\tau(S_{k+1}^\tau) = m_\rho(S_k^\rho)$; and, finally, $m_\tau(S_j^\tau) = m_\rho(S_{j-1}^\rho)$ for $k+2 \leq j \leq i$ (note also that, by the hypothesis, $m_\rho(S_i^\rho) = 0$). Since $\Pi_\rho(\{\omega\}) = \sum_{\omega \in S_i^\rho} m_\rho(S_j^\rho)$ (and similarly for Π_τ) we have then that
$\Pi_\tau(\{\omega\}) = \Pi_\rho(\{\omega\})$, $\forall \omega \notin S_{k+1}^\tau \setminus S_k^\tau$, while
$\Pi_\tau(\{\omega\}) = \Pi_\rho(\{\omega\}) - m_\tau(S_k^\tau)$, $\forall \omega \in S_{k+1}^\tau \setminus S_k^\tau$.

By Proposition 4, $m_\tau(S_k^\tau) \geq 0$: this completes the proof. □

Proposition 5 gives a criterion to identify possibilities which are not minimal with respect to inclusion among those generated by (2). The permutations which do not satisfy the hypothesis of Proposition 5 constitute the set of candidates to generate possibilities which are minimal with respect to inclusion, defined as:
$\mathcal{MC} = \{\rho :$ `if, for some` $k < i$,
$F_{\rho(i)} \subseteq S_k^\rho \wedge F_{\rho(i)} \not\subseteq S_{k-1}^\rho$
`then` $(F_{\rho(i)} \setminus S_{k-1}^\rho) = (S_k^\rho \setminus S_{k-1}^\rho)\}$.

We can now investigate the relationships between the possibilities generated by permutations in $\mathcal{MC}$ and those generated by (1).

Proposition 6 *For any permutation $\rho \in \mathcal{MC}$ there is a permutation σ of atoms such that $m_\rho(A) = m_\sigma(A)$, $\forall A \in 2^\Omega$.*

The proof can be obtained by building σ inductively. As for the basis case, considering the first set in ρ, S_1^ρ, by definition of $\mathcal{MC}$, it holds that: $\nexists F \in \mathcal{F} : F \subsetneq S_1^\rho$.
Therefore for any permutation σ such that $\omega_{\sigma(i)} \in S_1^\rho$, for $1 \leq i \leq |S_1^\rho|$ we have that $m_\sigma(E_j^\sigma) = 0$, for $1 \leq j < |S_1^\rho|$, $E_{|S_1^\rho|}^\sigma = S_1^\rho$ and $m_\sigma(E_{|S_1^\rho|}^\sigma) = m_\rho(S_1^\rho)$.
Now assume inductively that there are p and q ($1 \leq p \leq N$, $1 \leq q \leq |\mathcal{F}|$) such that:
$E_p^\sigma = S_q^\rho$ and $m_\sigma(E_p^\sigma) = m_\rho(S_q^\rho)$;
$\forall i, 1 \leq i < q, \exists j, 1 \leq j < p$ such that: $S_i^\rho = E_j^\sigma$ and $m_\rho(S_i^\rho) = m_\sigma(E_j^\sigma)$;
$\forall j, 1 \leq j < p$ such that $\nexists i, 1 \leq i < q$ such that $S_i^\rho = E_j^\sigma$, $m_\sigma(E_j^\sigma) = 0$.
If $S_q^\rho = \bigcup_{F \in \mathcal{F}} F$ (this happens in particular, but not necessarily only, when $q = |\mathcal{F}|$) the inductive hypothesis coincides with the thesis of the proposition. Otherwise let us show that the above conditions hold also for two indexes $p' > p$ and $q' > q$, for a suitable choice of the permutation σ. By definition of $\mathcal{MC}$, there is a $q' > q$ such that $S_q^\rho \subsetneq S_{q'}^\rho$, $S_i^\rho = S_q^\rho$, for $q \leq i < q'$, and for any focal set F such that $F \not\subseteq S_q^\rho \wedge F \subseteq S_{q'}^\rho$ it holds $(F \setminus S_q^\rho) = (S_{q'}^\rho \setminus S_q^\rho) = \Delta$. Let $p' = p + |\Delta|$. For any permutation σ such that $\omega_{\sigma(i)} \in \Delta$, for $p < i \leq p'$, we have that $m_\sigma(E_i^\sigma) = 0$, since there is no focal set included in E_i^σ but not in E_p^σ. Finally $E_{p'}^\sigma = S_{q'}^\rho$ and $m_\sigma(E_{p'}^\sigma) = m_\rho(S_{q'}^\rho)$, since $\{F \in \mathcal{F} : F \not\subseteq E_p^\sigma, F \subseteq E_{p'}^\sigma\} = \{F \in \mathcal{F} : F \not\subseteq S_q^\rho, F \subseteq S_{q'}^\rho\}$.
□

Conversely, it can be proved that among the possibilities generated by (1), only those corresponding to one of the possibilities generated by a permutation $\rho \in \mathcal{MC}$ are minimal.

Proposition 7 *For any permutation σ of atoms there is a permutation σ' such that: $\Pi_{\sigma'}(A) \leq \Pi_\sigma(A)$, $\forall A \in 2^\Omega$ and $\exists \tau \in \mathcal{MC}$, such that $\Pi_{\sigma'}(A) = \Pi_\tau(A)$, $\forall A \in 2^\Omega$.*

Let us first identify the conditions under which the mass assignment m_σ generated by a permutation σ of atoms coincides with the mass assignment m_ρ generated by a permutation ρ of focal elements. Let us denote by $\mathcal{F}^\sigma$ and $\mathcal{F}^\rho$ the sets of focal elements corresponding to m_σ and m_ρ. It must be the case that $\mathcal{F}^\sigma = \mathcal{F}^\rho$ and $\forall F^\sigma \in \mathcal{F}^\sigma, \forall F^\rho \in \mathcal{F}^\rho : F^\sigma = F^\rho \Rightarrow m(F^\sigma) = m(F^\rho)$. Without loss of generality, assume $\mathcal{F}^\sigma = \{F_1^\sigma, \ldots, F_h^\sigma\}$, with $F_i^\sigma \subsetneq F_j^\sigma$ if $i < j$ and, correspondingly, $\mathcal{F}^\rho = \{F_1^\rho, \ldots, F_h^\rho\}$, with $F_i^\sigma = F_i^\rho$, and $m(F_i^\sigma) = m(F_i^\rho)$, for $1 \leq i \leq h$. Recall that each F_i^σ corresponds to a set E_p^σ and let us define the function $s : [1 \ldots h] \rightarrow [1 \ldots N]$ such that $p = s(i)$ iff $E_p^\sigma = F_i^\sigma$. Clearly, $j > i \Rightarrow s(j) > s(i)$. Analogously, each F_i^ρ corresponds to a set S_q^ρ and, we define $r : [1 \ldots h] \rightarrow [1 \ldots |\mathcal{F}|]$ such that $q = r(i)$ iff $S_q^\rho = F_i^\rho$. Again, $j > i \Rightarrow r(j) > r(i)$, moreover, $F_1^\rho = S_1^\rho = F_{\rho(1)}$ and $F_h^\rho = S_{|\mathcal{F}|}^\rho = \bigcup_{F \in \mathcal{F}} F$. Letting $F_0^\sigma = F_0^\rho = \emptyset$ and $s(0) = r(0) = 0$, then for $1 \leq i \leq h$, it must be the case that $m_\sigma(E_j^\sigma) = 0$, for $s(i-1) < j < s(i)$, and also $m_\sigma(E_j^\sigma) = 0$, for $j > s(h)$. Similarly, for $1 \leq i < h$ it must be the case that $S_j^\rho = S_i^\rho$, for $r(i) < j < r(i+1)$.

Let us now show that for any mass assignment m_σ generated by a permutation σ of atoms there is another permutation of atoms σ^* such that $\Pi_{\sigma^*}(A) \leq \Pi_\sigma(A)$, $\forall A \in 2^\Omega$, and $m_{\sigma^*} = m_\rho$ for some permutation of focal elements ρ. This will prove the thesis, since, according to proposition 5, for any permutation ρ of focal elements there is another permutation τ respecting the desired conditions, and, according to proposition 6, there is a permutation σ' of atoms such that $m_{\sigma'} = m_\tau$.

We proceed step by step in the construction of σ^* and ρ. We will denote by σ_i^* the permutation produced at the i-th step. Each step is related to a focal set F_i^σ generated by the original permutation σ. The permutation σ_h^* produced at the (last) h-th step, in correspondence to F_h^σ, will coincide with the desired σ^*. Let us consider the smallest focal set of m_σ, namely F_1^σ (note that it does not necessarily coincide with a focal set of m). Let then assume the following definitions,

$\mathcal{FS}^1 = \{F \in \mathcal{F} : F \subseteq F_1^\sigma\}$, ($\mathcal{FS}^1$ is not empty by construction)

$g(1) = |\mathcal{FS}^1|$,

$FS_0 = \emptyset$, and $FS_1, \ldots, FS_{g(1)}$ a sequence of the elements of $\mathcal{FS}^1$ respecting the following conditions:

$\nexists F \in \mathcal{FS}^1 : F \subsetneq FS_1$;

$\nexists F \in \mathcal{FS}^1 : \emptyset \neq \{F \setminus \mathcal{US}_{i-1}^1\} \subsetneq \{FS_i \setminus \mathcal{US}_{i-1}^1\}$ for $1 < i \leq g(1)$,

where: $\mathcal{US}_j^1 = \bigcup_{1 \leq i \leq j} FS_i$, for $1 \leq j \leq g(1)$ and $\mathcal{US}_0^1 = \emptyset$.

To put it in other words, the sequence can be built by selecting for each i a set FS_i such that the set difference between FS_i and the union $\mathcal{US}_{i-1}^1$ of all previous sets in the sequence is minimal (with respect to set inclusion) among the differences generated by all other sets not yet included in the sequence.

A permutation σ_1^* can then be assigned following orderly the sequence $FS_1, \ldots, FS_{g(1)}$: for each FS_j, $\omega_{\sigma_1^*(i)}$ are selected in any order within $FS_j \setminus \mathcal{US}_{j-1}^1$ for $|\mathcal{US}_{j-1}^1| < i \leq (|\mathcal{US}_{j-1}^1| + |FS_j \setminus \mathcal{US}_{j-1}^1|)$. Considering the focal elements of the new permutation σ_1^* it can be noted that, by construction, $F_1^{\sigma_1^*} = \mathcal{US}_1^1$, $F_2^{\sigma_1^*} = \mathcal{US}_2^1$, but it may be the case that $\mathcal{US}_i^1 = \mathcal{US}_{i-1}^1$ for some $i > 2$ (this may happen when $(FS_j \setminus \mathcal{US}_{j-1}^1) = \emptyset$), therefore the sequences of elements $F_i^{\sigma_1^*}$ and $\mathcal{US}_i^1$ have not necessarily the same cardinality: the elements of $F_i^{\sigma_1^*}$ (which are all distinct) are in one-to-one correspondence with the distinct elements within the sequence of $\mathcal{US}_i^1$. Moreover, it may be the case that $\mathcal{US}_{g(1)}^1 \neq F_1^\sigma$, namely that $\exists \omega \in F_1^\sigma : \omega \notin FS_i$ for $1 \leq i \leq g(1)$. In that case, $\omega_{\sigma_1^*(i)}$ are selected in any order within $(F_1^\sigma \setminus \mathcal{US}_{g(1)}^1)$ for $|\mathcal{US}_{g(1)}^1| < i \leq |F_1^\sigma|$.

Letting then $\omega_{\sigma_1^*(i)} = \omega_{\sigma(i)}$ for $i > |F_1^\sigma|$ it holds that $\Pi_{\sigma_1^*}(\{\omega\}) \leq \Pi_\sigma(\{\omega\})$, $\forall \omega \in \Omega$. In fact, by construction $\Pi_{\sigma_1^*}(\{\omega\}) = \Pi_\sigma(\{\omega\})$,

$\forall\omega \notin F_1^\sigma$, while $\forall\omega \in F_1^\sigma$ let us denote $mni^1(\omega) = \max\{i : \omega \notin \mathcal{US}_i^1\}$ i.e. the maximum index i such that ω is not included in the corresponding $\mathcal{US}_i^1$. By construction, $0 \le mni^1(\omega) \le g(1)$ and:
$\Pi_\sigma(\{\omega\}) - \Pi_{\sigma_1^*}(\{\omega\}) = \sum_{F \subseteq \mathcal{US}^1_{mni^1(\omega)}} m(F) = C(\mathcal{US}^1_{mni^1(\omega)}) \ge 0$.
Up to the first $|\mathcal{US}^1_{g(1)}|$ atoms, the permutation σ_1^* satisfies the desired requirement. The corresponding ρ is given by the sequence used in the construction of σ_1^*: $F_{\rho(i)} = FS_i$, for $1 \le i \le g(1)$. If F_1^σ is the only focal set of m_σ, we are done. Otherwise, let us examine how a generic σ_i^* can be obtained from σ_{i-1}^*.

To this purpose we extend the notation introduced above as follows:
$\mathcal{FS}^i = \{F \in \mathcal{F} : F \not\subseteq F_{i-1}^\sigma, F \subseteq F_i^\sigma\}$,
$g(i) = |\mathcal{FS}^i| + g(i-1)$,
$FS_{1+g(i-1)}, \ldots, FS_{g(i)}$ a sequence of the elements of $\mathcal{FS}^i$ such that for $g(i-1) < j \le g(i)$:
$\nexists F \in \mathcal{FS}^i : \emptyset \neq \{F \setminus \mathcal{US}^i_{j-1}\} \subsetneq \{FS_j \setminus \mathcal{US}^i_{j-1}\}$
where
$\mathcal{US}^i_j = \bigcup_{1 \le m \le j} FS_m$, for $g(i-1) \le j \le g(i)$.
Note in particular that $\mathcal{US}^i_{g(i-1)} = \mathcal{US}^{i-1}_{g(i-1)}$.
In words, the sequence $FS_{1+g(i-1)}, \ldots, FS_{g(i)}$ can be built by selecting initially within $\mathcal{FS}^i$ a set $FS_{1+g(i-1)}$ such that its set difference with respect to $\mathcal{US}^{i-1}_{g(i-1)} = \mathcal{US}^i_{g(i-1)}$ is minimal, then computing $\mathcal{US}^i_{g(i-1)+1}$ and repeating, among those left in $\mathcal{FS}^i$, the choice of a set such that its difference with respect to the upgraded $\mathcal{US}^i_j$ is minimal. Define also $ulevel(i) = |\mathcal{US}^i_{g(i)}|$, which corresponds to the index reached in the construction of permutation σ^* at the i-th step. Finally, note that the index reached in the construction of permutation ρ at the i-th step is equal to $g(i)$.
A permutation σ_i^* can then be assigned as:
for $1 \le j \le ulevel(i-1)$: $\omega_{\sigma_i^*(j)} = \omega_{\sigma_{i-1}^*(j)}$;
for $ulevel(i-1) < j \le ulevel(i)$, the sequence $FS_{1+g(i-1)}, \ldots, FS_{g(i)}$ is followed orderly: for each FS_k, $\omega_{\sigma_i^*(j)}$ are selected in any order within $(FS_k \setminus \mathcal{US}^i_{k-1})$ for $|\mathcal{US}^i_{k-1}| < j \le (|\mathcal{US}^i_{k-1}| + |FS_k \setminus \mathcal{US}^i_{k-1}|)$;
for $|\mathcal{US}^i_{g(i)}| < j \le |F_i^\sigma|$, $\omega_{\sigma_i^*(j)}$ are selected in any order within $(F_i^\sigma \setminus \mathcal{US}^i_{g(i)})$;
for $j > |F_i^\sigma|$: $\omega_{\sigma_i^*(j)} = \omega_{\sigma_{i-1}^*(j)} = \omega_{\sigma(j)}$.
In words, σ_i^* is built on σ_{i-1}^* by possibly changing the positions only of the atoms included in $F_i^\sigma \setminus \mathcal{US}^{i-1}_{g(i-1)}$, analogously to the basis case.
Again, $\Pi_{\sigma_i^*}(\{\omega\}) \le \Pi_{\sigma_{i-1}^*}(\{\omega\})$, $\forall\omega \in \Omega$: by construction $\Pi_{\sigma_i^*}(\{\omega\}) = \Pi_{\sigma_{i-1}^*}(\{\omega\})$, $\forall\omega \notin (F_i^\sigma \setminus \mathcal{US}^{i-1}_{g(i-1)})$, while $\forall\omega \in (F_i^\sigma \setminus \mathcal{US}^{i-1}_{g(i-1)})$ let us denote $mni^i(\omega) = \max\{j : \omega \notin \mathcal{US}^i_j\}$ i.e. the maximum index j such that ω is not included in the corresponding $\mathcal{US}^i_j$. By construction, $g(i-1) \le mni^i(\omega) \le g(i)$
Then, for these atoms, it holds that
$\Pi_{\sigma_{i-1}^*}(\{\omega\}) - \Pi_{\sigma_i^*}(\{\omega\}) =$
$\sum_{F \not\subseteq \mathcal{US}^{i-1}_{g(i-1)}} m(F) - \sum_{F \not\subseteq \mathcal{US}^i_{mni^i(\omega)}} m(F) =$
$\sum_{F \not\subseteq \mathcal{US}^{i-1}_{g(i-1)}, F \subseteq \mathcal{US}^i_{mni^i(\omega)}} m(F) =$
$C(\mathcal{US}^i_{mni^i(\omega)}) - C(\mathcal{US}^{i-1}_{g(i-1)}) \ge 0$ The permutation σ_i^* extends up to the first $|\mathcal{US}^i_{g(i)}|$ atoms, the satisfaction of the requirements. The corresponding ρ is defined by the sequence: $F_{\rho(i)} = FS_i$, for $1 \le i \le g(i)$.$\square$
To complete the characterization of minimal outer approximations we finally need to prove that all approximations generated by a permutation $\rho \in \mathcal{MC}$ are minimal.

Proposition 8 *For any permutation $\rho \in \mathcal{MC}$, $\nexists\tau \in \mathcal{MC} : \Pi_\tau \neq \Pi_\rho$ and Π_τ is dominated by or dominates Π_ρ.*

First note that two permutations ρ and τ in $\mathcal{MC}$ give rise to different approximating possibilities only if $\exists i$: $S_i^\rho \neq S_i^\tau$. Consider the minimum such index i, i.e. assume $S_j^\rho = S_j^\tau$ for $0 \le j < i$, and $S_i^\rho \neq S_i^\tau$. Then $F_{\rho(i)} \neq F_{\tau(i)}$, moreover $F_{\rho(i)} = F_{\tau(j)}$ for some $j > i$ and similarly $F_{\tau(i)} = F_{\rho(k)}$ for some $k > i$. By the properties of $\mathcal{MC}$, then $(F_{\rho(i)} \setminus S^\rho_{i-1}) \not\subseteq (F_{\tau(i)} \setminus S^\rho_{i-1})$ and, vice versa, $(F_{\tau(i)} \setminus S^\rho_{i-1}) \not\subseteq (F_{\rho(i)} \setminus S^\rho_{i-1})$. We have therefore $(F_{\rho(i)} \setminus F_{\tau(i)}) \neq \emptyset$ and $(F_{\tau(i)} \setminus F_{\rho(i)}) \neq \emptyset$. Consider now an atom $\omega \in (F_{\rho(i)} \setminus F_{\tau(i)})$ and an atom $\omega' \in (F_{\tau(i)} \setminus F_{\rho(i)})$. Then, $\Pi_\rho(\{\omega\}) > \Pi_\rho(\{\omega'\})$, since $\Pi_\rho(\{\omega\}) = \sum_{F \not\subset S^\rho_{i-1}} m(F) > \sum_{F \not\subset S^\rho_i} m(F) \ge \Pi_\rho(\{\omega'\})$ and, in the same way, $\Pi_\tau(\{\omega'\}) > \Pi_\tau(\{\omega\})$. Moreover, $\Pi_\rho(\{\omega\}) = \sum_{F \not\subset S^\rho_{i-1}} m(F) = \Pi_\tau(\{\omega'\})$. By suitable substitutions in the above inequalities, we obtain: $\Pi_\rho(\{\omega\}) > \Pi_\tau(\{\omega\})$ and $\Pi_\tau(\{\omega'\}) > \Pi_\rho(\{\omega'\})$.$\square$

6 Conclusions

Let us summarize the results presented in the paper. Denote $outer(C)$ the set of consonant outer approximations of a capacity C defined on 2^Ω and $minouter(C)$ the minimal outer approximations of C. Let Σ be the set of all permutations σ of atoms of Ω and $N_\Sigma(C)$ the set of all the corresponding necessity measures generated by (1). Similarly, let R be the set of all permutations ρ of focal elements of C and $N_R(C)$ the set of all the corresponding necessity measures generated by (2). Finally, let $N_{\mathcal{MC}}(C)$ be the set of necessities generated by permutations in $\mathcal{MC}$. Then:
$N_\Sigma(C) \subseteq outer(C)$, by Proposition 2;
$minouter(C) \subseteq N_\Sigma(C)$, by Proposition 3;
$N_R(C) \subseteq outer(C)$, by Proposition 4;
$N_R(C) \cap minouter(C) \subseteq N_{\mathcal{MC}}$, by Proposition 5;
$N_{\mathcal{MC}}(C) \subseteq N_\Sigma(C)$, by Proposition 6;
$N_\Sigma(C) \cap minouter(C) \subseteq N_{\mathcal{MC}}(C)$, by Proposition 7, and since $minouter(C) \subseteq N_\Sigma(C)$, this implies $minouter(C) \subseteq N_{\mathcal{MC}}(C)$;
by Proposition 8, $N_{\mathcal{MC}}(C) \subseteq minouter(C)$, and this finally gives the desired characterization: $N_{\mathcal{MC}}(C) = minouter(C)$.

Apart from their theoretical interest, these results are applicable in the field of multi-agent systems, since they provide a basis for information interchange among heterogeneous software components [1]. In particular, the proposed generalization applies to the case of coherent imprecise probabilities [11], whose consonant approximation has not been considered yet in the literature. An analysis of the quality of the approximation produced is among the directions of future work. In particular, while all elements of $N_{\mathcal{MC}}(C)$ are minimal with respect to inclusion, they are not necessarily equivalent as far as additional imprecision is concerned (as also pointed out in [7] for belief functions). This issue, and the relevant problem of defining algorithms for computing approximations minimizing imprecision, will be considered in the next future.

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