# Shortfall-dependant Risk Measures (and Previsions)

**Pietro Baroni**
Dip. Elettronica per l'Automazione
Univ. of Brescia, Italy
baroni@ing.unibs.it

**Renato Pelessoni**
Dip. Mat. Appl. "B. de Finetti"
Univ. of Trieste, Italy
renato.pelessoni@econ.units.it

**Paolo Vicig**
Dip. Mat. Appl. "B. de Finetti"
Univ. of Trieste, Italy
paolo.vicig@econ.units.it

## Abstract

Because of their simplicity, risk measures are often employed in financial risk evaluations and related decisions. In fact, the risk measure $\rho(X)$ of a random variable $X$ is a real number customarily determining the amount of money needed to face the potential losses $X$ might cause. At a sort of second-order level, the adequacy of $\rho(X)$ may be investigated considering the part of the losses it does not cover (its shortfall). This may suggest employing a further, more prudential risk measure, taking the shortfall of $\rho(X)$ into account. In this paper a family of shortfall-dependant risk measures is proposed, investigating its consistency properties and its utilization in insurance pricing. These results are obtained and subsequently extended within the framework of imprecise previsions, of which risk measures are an instance. This also leads us to investigate properties of a rather weak consistency notion for imprecise previsions, termed 1–convexity.

**Keywords:** Risk Measures, Imprecise Previsions, Shortfall.

## 1 Introduction

Risk measures are an important and widely studied tool in the field of financial evaluations and decisions. In spite of their significant relationships with other uncertainty theories, in particular with precise and imprecise previsions [2, 14], steps towards bridging the gap and enabling cross-fertilization between these research areas have been taken only relatively recently (see for instance [8, 9, 13]). In this paper we contribute to this effort with focus on *shortfall-dependant* risk measures, and, in particular, on the so-called *Dutch risk measures.*

To give a quick idea of the whole context, we first recall that, among its interpretations, the risk measure $\rho(X)$ of a random variable $X$ supplies a real number determining how much money should be reserved to face potential losses arising from $X$. Several kinds of *consistency* requirements have been considered in the literature in order to ensure that $\rho(X)$ is "sound" (in some sense). Defining (and satisfying) these requirements is particularly important (and not so obvious) when the domain of a risk measure is a *set* of random quantities. Consistency requirements, in general, allow more alternative risk measures. Selecting a specific measure is, in a sense, arbitrary and may reflect the risk attitude or other subjective features of the agent evaluating the risk. For instance, a coherent but extremely cautious choice for $\rho(X)$ is $\rho(X) = -\inf X$. Clearly, this is questionable from another point of view, since it is likely to reserve an excessive amount of money. For this reason, it is rather common (and reasonable) that the chosen risk measure does not entirely cover all potential losses.

The amount of losses not covered by a risk measure is called its *shortfall.* Given a random variable $X$ and a risk measure $\rho$, the shortfall is a non-negative random variable which is a function of $X$ and $\rho(X)$ and quantifies the possible losses exceeding $\rho(X)$. Given a risk measure, a derived risk measure can then be defined which depends on the original one by taking account of its shortfall in some way. Examples of motivations for defining this kind of "second-order" risk measure (which can be regarded as an adjustment of the first one) will be better discussed in Section 3.3. A significant class of shortfall-dependant risk measures is represented by Dutch risk measures, which can be given a practically important interpretation in the domain of insurance pricing.

This paper provides some new definitions and results concerning shortfall-dependant risk measures and previsions and their consistency properties.

After recalling basic concepts and results about previsions and risk measures in Section 2, we define in Section 3 a generalized family of Dutch risk measures, relate it to a family of lower previsions and investigate its consistency properties. In particular, we show that this family of measures preserves the consistency property of *coherence* (alternatively *convexity*) provided that the original measure and the measure evaluating its shortfall satisfy it. In insurance pricing, generalized Dutch risk measures allow (unlike previous proposals) certain premium policies (double loading), while preserving the above mentioned properties; this is discussed in Section 3.3. In Section 4 we provide a result concerning the ability of "Dutch-like" imprecise previsions to preserve the weaker consistency property of *1-convexity.* In this context, we also explore some properties of 1-convexity, including its close relationships with the notions of capacity and niveloid.

## 2 Preliminaries

### 2.1 Precise and imprecise previsions

Let $D$ be an arbitrary (non-empty) set of *bounded* random variables (unbounded variables will not be considered in this paper). We shall use the term *prevision* to denote a mapping $P : D \to \mathbb{R}$ which is understood to be, unless otherwise stated, a *coherent* (precise) prevision in the sense of de Finetti [2]. As well known, this means that $P$ satisfies a certain no-arbitrage condition in an idealized betting scheme. For each $X \in D$, $P(X)$ "summarizes" $X$, also meaning that whenever an expectation $E(X)$ is given, $P(X) = E(X)$.

In many practical situations it may be more appropriate to assess an imprecise evaluation on each $X \in D$: a *lower* ($\underline{P}$) or an *upper* ($\overline{P}$) prevision. A precise prevision, coherent or not, corresponds to the special case $\overline{P}(X) = \underline{P}(X) = P(X)$, $\forall X \in D$. The upper (lower) prevision $\overline{P}(X)$ ($\underline{P}(X)$) for $X$ has been given in [14] the meaning of infimum selling price (supremum buying price) for $X$; this interpretation is relevant also in relating imprecise previsions and risk measures [8]. Consistency requirements for imprecise previsions were proposed in [9, 14] by modifying de Finetti's betting scheme. In all instances, it is sufficient to refer to either lower or upper previsions only (on $D$ or on $D^- = \{X : -X \in D\}$, respectively) because of the conjugacy equality

$$\overline{P}(X) = -\underline{P}(-X). \tag{1}$$

In particular, *coherent* lower previsions may be defined as follows [14]:

**Definition 1** $\underline{P}$ *is a* coherent lower prevision *on $D$ iff $\forall n \in \mathbb{N}^+$, $\forall X_0, \ldots, X_n \in D$, $\forall s_0, \ldots, s_n \geq 0$, defining $\underline{G} = \sum_{i=1}^n s_i(X_i - \underline{P}(X_i)) - s_0(X_0 - \underline{P}(X_0))$, it holds that $\sup \underline{G} \geq 0$.*

The weaker notion of lower prevision that avoids sure loss [14] may be regarded as a minimal consistency requirement. The concept of *centered convex* (or *C-convex*) lower prevision was introduced in [9], and is somewhat intermediate between those of lower prevision that is coherent and that avoids sure loss. The definition of the still weaker notion of convexity is obtained from Definition 1 by adding there the constraint $\sum_{i=1}^n s_i = s_0$.

**Definition 2** $\underline{P}$ *is a* convex lower prevision *on $D$ iff $\forall n \in \mathbb{N}^+$, $\forall X_0, \ldots, X_n \in D$, $\forall s_0, \ldots, s_n \geq 0$ such that $\sum_{i=1}^n s_i = s_0$, defining $\underline{G} = \sum_{i=1}^n s_i(X_i - \underline{P}(X_i)) - s_0(X_0 - \underline{P}(X_0))$, it holds that $\sup \underline{G} \geq 0$.*

*A convex lower prevision such that ($0 \in D$ and) $\underline{P}(0) = 0$ is* C-convex.

Recalling that, given an event $A$, its lower probability $\underline{P}(A)$ is interpreted as the lower prevision of the indicator function of $A$, the extra assumption $\underline{P}(0) = 0$ is quite natural: $\underline{P}(0)$ represents just the lower uncertainty evaluation we would assign to the impossible event $\emptyset$.

Referring to [2, 9, 14] for a detailed study of the notions presented in this subsection, we recall now some results for later use:

**Proposition 1** *Let $\mu : D \to \mathbb{R}$.*

*a) Let $\mu$ be a prevision (alternatively, a coherent, convex or C-convex lower prevision). Whatever is $D' \supset D$, there exists an extension of $\mu$ on $D'$ which is a prevision (alternatively, which is a coherent, convex or C-convex lower prevision, respectively).*

*b) When $D$ is a linear space, $\mu$ is a coherent lower prevision on $D$ iff:*

- *b1) $\mu(X) \geq \inf X$, $\forall X \in D$*
- *b2) $\mu(\lambda X) = \lambda\mu(X)$, $\forall X \in D$, $\lambda > 0$*
- *b3) $\mu(X + Y) \geq \mu(X) + \mu(Y)$, $\forall X, Y \in D$*

*c) When $D$ is a linear space containing real constants, $\mu$ is a convex lower prevision on $D$ iff:*

- *c1) $\mu(X+k) = \mu(X)+k$, $\forall X \in D$, $\forall k \in \mathbb{R}$ (translation invariance)*
- *c2) $\forall X, Y \in D$, if $X \leq Y$ then $\mu(X) \leq \mu(Y)$ (monotonicity)*
- *c3) $\mu(\lambda X + (1-\lambda)Y) \geq \lambda\mu(X) + (1-\lambda)\mu(Y)$, $\forall X, Y \in D$, $\forall \lambda \in [0, 1]$ (concavity)*

*d) If $\mu_1$ and $\mu_2$ are both coherent (alternatively, convex or C-convex) lower previsions then $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$,*

$\lambda \in [0,1]$ *is a coherent (alternatively, convex or C-convex) lower prevision.*

It ensues from Proposition 1 that the conditions in c), in particular c1) and c2), are necessary for convexity of a lower prevision $\underline{P}$ even when $D$ is not a linear space but the relevant quantities are well-defined. In fact, if $\underline{P}$ is convex it allows for a convex extension on a linear space $\mathcal{L} \supset D$ where c1) and c2) must hold. Since coherence implies convexity, c1) and c2) hold for coherent lower previsions, and for previsions too. It can be shown that b1) (but generally not b2), nor b3)) holds for any C-convex $\underline{P}$. Precise previsions are linear: $P(X+Y) = P(X) + P(Y)$ whenever $X, Y, X+Y \in D$. Using (1), the results of Proposition 1 can be easily restated in their specular version for upper previsions.

Coherent imprecise previsions are characterized as envelopes of (precise) previsions, by means of envelope theorems [14]. We recall the result for coherent upper previsions:

**Proposition 2** *Let* $\overline{P} : D \to \mathbb{R}$, $\mathcal{M} = \{P : P(X) \leq \overline{P}(X), \forall X \in D, P$ *is a prevision on* $D\}$. *Then* $\overline{P}$ *is a coherent upper prevision on* $D$ *iff* $\overline{P}(X) = \max_{P \in \mathcal{M}}\{P(X)\}, \forall X \in D$.

Generalizations of Proposition 2 characterize similarly convex and C-convex previsions [9].

### 2.2 Risk measures

The risk measure $\rho(X)$ of a random variable $X$ is a real number measuring how "risky" $X$ is. Usually $X$ is the future value, in some currency, of a financial asset and $\rho(X)$ corresponds to the amount of money to be reserved to cover losses potentially arising from (negative values of) $X$. When $\rho(X) < 0$, this means that we could add $\rho(X)$ to $X$, i.e. subtract $|\rho(X)|$ from $X$, keeping the resulting $X - |\rho(X)|$ desirable (no reserving is believed to be necessary to cover risks from $X - |\rho(X)|$). A risk measure $\rho$ on $D$ is thus a real map $\rho : D \to \mathbb{R}$. A risk measure is a relatively simple tool to take basic financial decisions, and this explains the popularity of such instruments in risk theory and practice. Hence risk measures have been extensively studied, but their relationship with other uncertainty theories was mostly overlooked. It was shown in [8] that the risk measure $\rho(X)$ can be interpreted as an upper prevision for $-X$, being the infimum price one would ask to shoulder $X$, or to sell $-X$. Because of this and (1), the following fundamental equality holds

$$\rho(X) = \overline{P}(-X) = -\underline{P}(X) \quad (2)$$

This equality lets us transfer results from risk measures to upper or lower previsions or vice versa. In Section 3 most statements are given for risk measures, but proven in their corresponding version for lower previsions, following the prevailing custom in the relevant literature. Given equation (2), the consistency notions of coherence, convexity and C-convexity are easily reworded for risk measures [8, 9]. Thus, for instance:

**Definition 3** $\rho : D \to \mathbb{R}$ *is a* coherent risk measure *on* $D$ *iff* $\forall n \in \mathbb{N}^+, \forall X_0, \ldots, X_n \in D, \forall s_0, \ldots, s_n \geq 0$, *defining* $G_\rho = \sum_{i=1}^n s_i(X_i + \rho(X_i)) - s_0(X_0 + \rho(X_0))$, *it holds that* $\sup G_\rho \geq 0$.

Definition 3 includes as a special case the notion of coherent risk measure defined in [1] through a set of axioms and assuming that the domain is a linear space. Analogously, the concept of convex risk measure [9], obtained adding condition $\sum_{i=1}^n s_i = s_0$ in Definition 3, generalizes to arbitrary domains a notion developed in [5] for linear spaces only. For an overview of the many interactions between imprecise previsions and risk measures, see [13].

## 3 Shortfall-based risk measures

Whatever the risk measure $\rho$ is, it might be inadequate to fully cover losses. Suppose for instance we assess a priori $\rho(X) = 5$ while a posteriori $X$ assumes the value $-8$: $\rho$ covers only partly the loss arising from $X$, since there remains a *residual loss* or *shortfall* of 3 in *absolute value*, after employing the reserve money of 5. If instead, a posteriori, $X = -2$, the protection ensured by $\rho(X) = 5$ is full and the shortfall assumes the value 0.

Formally, given a random variable $X$ and a risk measure $\rho(X)$, the *shortfall* of $\rho(X)$ is the random variable $\max(-\rho(X) - X, 0)$.

In the following we will use the shortened notation $(Y)_+ \triangleq \max(Y, 0)$ where $Y$ is a random variable. Accordingly the shortfall will be denoted as $(-\rho(X) - X)_+$. We will also use the dual notation $(Y)_- \triangleq \min(Y, 0)$.

In this paper we focus on risk measures which take account of the shortfall arising from a previously assessed risk measure. More specifically, we shall generalize the family of Dutch risk measures.

### 3.1 Dutch risk measures

Suppose a (precise) prevision $P_0$ is assessed on $D$. We call *Dutch risk measure* the measure

$$\rho_D(X) = P_0(-X) + cP_1[(P_0(X) - X)_+], c \in [0,1] \quad (3)$$

where $P_1$ is a prevision on a set $D_1$ such that (3) is well-defined (in particular the set $D_1$ must include the random variables $(P_0(X) - X)_+$, $\forall X \in D$). Since $P_0(X) - X = -P_0(-X) - X$, $(P_0(X) - X)_+$ is the shortfall arising from using the prevision $P_0$ as a risk measure for $X$ ($\rho(X) = P_0(-X)$). It is intuitively clear that this choice for $\rho$ is inadequate since a risk measure should be typically asymmetric, giving higher weight to lower values of $X$. However, $P_0$ can be taken as a basis for building a more appropriate risk measure. The new risk measure takes account of the former one through prevision $P_1$, which evaluates the size of $\rho$'s shortfall. Thus $P_1$ should typically be assessed *independently* of $P_0$, at a later stage and on a possibly different domain $D_1$. The measure $\rho_D(X)$ is coherent: a direct proof may be found in [13]. An earlier version of (3), to be discussed in Section 3.3, appeared in [6] and later in [3, 7]. The measures discussed in these papers may be written as

$$\rho'_D(X) = E(-X) + cE[(dE(X) - X)_+], c \in ]0,1], d > 0 \quad (4)$$

In (4), $P_0$ and $P_1$ are replaced by an expectation. It was shown in [3] that if $c = d = 1$, $\rho'_D$ is defined on a linear space, and the expectations are computed with respect to a *common* probability measure, then $\rho'_D$ is coherent.

A distinguishing feature of (3) with respect to (4) is its emphasizing that the uncertainty evaluations $P_0$ and $P_1$ could be assessed independently, while this is not possible in (4) if the same underlying probability is used to compute all expectations. To highlight the relevance of this distinction, let us consider the following extreme example.

*Example.* Let $D = \{X\}$, $X \leq 0$ and assign $\rho(X) = P_0(-X) = P_0(X) = 0$. This is a coherent but highly unbalanced choice: no reserve money is required in a case where no gain is possible, whatever value $X$ will have. Here $(P_0(X) - X)_+ = -X$, hence using (3) we may correct the evaluation if $P_1[(P_0(X) - X)_+] = P_1(-X) > 0$. However no correction is possible if we require that $P_0 = P_1$, $D = D_1$, since then $P_1(-X) = P_0(-X) = 0$.

### 3.2 Generalized Dutch risk measures

We introduce now a new family of risk measures, which generalizes the risk measures in (3) in a twofold way. First, a natural idea is to replace $P_0(-X)$ with a risk measure $\rho(X)$. Further, we might be unable to precisely evaluate the shortfall $(-\rho(X) - X)_+$, therefore $P_1$ could be substituted by an imprecise evaluation: given that the new risk measure, say $\rho_c(X)$, should be a prudential correction of $\rho(X)$, an upper prevision $\overline{P}$ seems more appropriate than a lower one. We therefore propose

$$\rho_c(X) = \rho(X) + c\overline{P}[(-\rho(X) - X)_+], c \in [0,1] \quad (5)$$

What are the consistency properties of $\rho_c(X)$? We shall now prove the following proposition.

**Proposition 3** *Let $\rho$ be a coherent risk measure on $D$, $\overline{P}$ a coherent upper prevision on a set $D_U$ such that (5) is well-defined . Then $\rho_c(X)$ as defined by (5) is a coherent risk measure on $D$.*

We shall prove Proposition 3 in its corresponding version for lower previsions which, using (2) and elementary properties of max and min, is stated as follows:

**Proposition 4** *Let $\underline{P}_1$, $\underline{P}_2$ be two coherent lower previsions on $D_1$, $D_2 \supset \{Y : Y = \min(X + h, k), X \in D_1, h, k \in \mathbb{R}\}$ respectively. Then*

$$\underline{P}_c(X) = \underline{P}_1(X) + c\underline{P}_2[(X - \underline{P}_1(X))_-], c \in [0,1] \quad (6)$$

*is a coherent lower prevision on $D_1$.*

The proof relies on the following Lemma.

**Lemma 1** *Given $\underline{P}_1$, $\underline{P}_2$ as in Proposition 4,*

$$\underline{P}^*(X) = \underline{P}_2[\min(X, \underline{P}_1(X))] \quad (7)$$

*is a coherent lower prevision on $D_1$.*

**Proof.** By Proposition 1,a), there exist coherent lower previsions extending, respectively, $\underline{P}_1$ and $\underline{P}_2$ on some linear space $\mathcal{L} \supset D_1 \cup D_2$. Using such extensions and (7), $\underline{P}^*$ may be extended on $\mathcal{L}$ too. Consider one such extension (also named $\underline{P}^*$): if it is coherent on $\mathcal{L}$, its restriction on $D_1$ (our starting $\underline{P}^*$) is coherent too.

Coherence of $\underline{P}^*$ on $\mathcal{L}$ may be proved by checking the axioms in Proposition 1,b). Recall for this that (the extensions of) $\underline{P}_1$ and $\underline{P}_2$, being coherent on $\mathcal{L}$, satisfy all axioms listed in Proposition 1, b) and c).

To check b1) for $\underline{P}^*$, we apply b1) to $\underline{P}_1$, c2) to $\underline{P}_2$ and property $\underline{P}_2(k) = k, \forall k \in \mathbb{R}$ ([14], sec. 2.6.1,(b)). Then $\underline{P}^*(X) = \underline{P}_2[\min(X, \underline{P}_1(X))] \geq \underline{P}_2[\min(X, \inf X)] = \underline{P}_2(\inf X) = \inf X$.

To check b2) for $\underline{P}^*$, apply b2) to $\underline{P}_1$, $\underline{P}_2$: $\underline{P}^*(\lambda X) = \underline{P}_2[\min(\lambda X, \underline{P}_1(\lambda X))] = \underline{P}_2[\lambda \min(X, \underline{P}_1(X))] = \lambda \underline{P}^*(X)$, $\forall \lambda > 0$.

Finally we check b3) for $\underline{P}^*$, using b3), c2) and property

$$\min(a + b, c + d) \geq \min(a, c) + \min(b, d). \quad (8)$$

Then, $\underline{P}^*(X + Y) = \underline{P}_2[\min(X + Y, \underline{P}_1(X + Y))] \geq \underline{P}_2[\min(X + Y, \underline{P}_1(X) + \underline{P}_1(Y))] \geq \underline{P}_2[\min(X, \underline{P}_1(X))] + \underline{P}_2[\min(Y, \underline{P}_1(Y))] = \underline{P}^*(X) + \underline{P}^*(Y)$. □

**Proof of Proposition 4**. Using, at the second equality, c1) (with $k = \underline{P}_1(X)$) and property $\min(f, 0) + k =$

$\min(f+k,k)$, we can write (6) as follows: $\underline{P}_c(X) = (1-c)\underline{P}_1(X) + c\underline{P}_1(X) + c\underline{P}_2[(X - \underline{P}_1(X))_-] = (1-c)\underline{P}_1(X) + c\underline{P}_2[(X-\underline{P}_1(X))_- + \underline{P}_1(X)] = (1-c)\underline{P}_1(X) + c\underline{P}_2[\min(X,\underline{P}_1(X))] = (1-c)\underline{P}_1(X) + c\underline{P}^*(X)$. Hence $\underline{P}_c$ is coherent, by Proposition 1,d). □

The result in Proposition 3 can be further generalized to the case of convex or C-convex $\rho$ and $\overline{P}$.

**Proposition 5** *Let $\rho$ be a convex risk measure on $D$, $\overline{P}$ a convex upper prevision on $D_1$. Then $\rho_c(X)$ defined by (5) is a convex risk measure. If $\rho$ and $\overline{P}$ are C-convex, $\rho_c(X)$ is C-convex too.*

**Proof.** The proof resembles that of Proposition 3: we prove that if $\underline{P}_1$ and $\underline{P}_2$ are convex $\underline{P}_c(X)$ in (6) is convex too, by preliminarily proving that $\underline{P}^*(X)$ in (7) is convex. Much like the proof of Lemma 1, we can check convexity of an extension of $\underline{P}^*$ on a linear space containing real constants $\mathcal{L} \supset D_1 \cup D_2$. This is tantamount to verifying the axioms in Proposition 1,c) for the extended $\underline{P}^*$.

As for c1), we get $\underline{P}^*(X+k) = \underline{P}_2[\min(X+k, \underline{P}_1(X+k))] = \underline{P}_2[\min(X,\underline{P}_1(X)) + k] = \underline{P}^*(X) + k$.

To prove c2), let $X \leq Y$. Then $\underline{P}_1(X) \leq \underline{P}_1(Y)$, $\min(X,\underline{P}_1(X)) \leq \min(Y,\underline{P}_1(Y))$ and c2) follows from monotonicity of $\underline{P}_2$, which implies $\underline{P}_2[\min(X,\underline{P}_1(X))] \leq \underline{P}_2[\min(Y,\underline{P}_1(Y))]$.

To prove c3), apply: c3) itself, properties of min (including (8)) and c2), getting: $\underline{P}^*(\lambda X + (1-\lambda)Y) = \underline{P}_2[\min(\lambda X + (1-\lambda)Y, \underline{P}_1(\lambda X + (1-\lambda)Y))] \geq \underline{P}_2[\min(\lambda X + (1-\lambda)Y, \lambda\underline{P}_1(X) + (1-\lambda)\underline{P}_1(Y))] \geq \underline{P}_2[\lambda \min(X,\underline{P}_1(X)) + (1-\lambda)\min(Y,\underline{P}_1(Y))] \geq \lambda\underline{P}^*(X) + (1-\lambda)\underline{P}^*(Y)$.

Having thus established that $\underline{P}^*$ is convex, we write, as in the proof of Proposition 4,

$$\underline{P}_c(X) = (1-c)\underline{P}_1(X) + c\underline{P}^*(X). \tag{9}$$

We can do this because the only property of imprecise previsions exploited in the derivation of (9) is c1), which holds for convex lower previsions too. From (9), convexity of $\underline{P}_c$ is immediate using Proposition 1,d). Finally, it is trivial to see that if $\underline{P}_1$ and $\underline{P}_2$ are C-convex then $\underline{P}_c(0) = 0$. □

### 3.3 Implications for insurance pricing

From the preceding subsection we know that (5) can be employed to get a sort of "second-order" risk measure $\rho_c(X)$ from a previously assessed $\rho(X)$, taking account of the potential inadequacy of $\rho(X)$ to cover residual losses. The measure $\rho_c(X)$ is coherent, alternatively convex, if $\rho(X)$ and $\overline{P}$ are so. There may be many reasons for applying (5): for instance, the use of $\rho(X)$ may be imposed by some regulatory authority but an agent may wish to consider a different, even more prudential measure for certain purposes. Or, conversely, it is the regulatory authority that computes $\rho_c(X)$ on the basis of its own evaluation $\overline{P}$ of the shortfall of the measure $\rho$ adopted by the firm management. This situation is not uncommon, since the management may tend to reserve little money, favouring more profitable (and risky) investments.

To explore yet another interpretation of (5), recall that $\rho(X)$ has the meaning of the infimum price an agent would ask to shoulder $X$ [8], and suppose now $X \leq 0$. This is not unusual in insurance, where the insurer asks for a premium to run the risk of paying $-X \geq 0$. Here $\rho(X)$ represents the premium and a rule for determining it is named *premium principle*. A common procedure to obtain a premium principle starts from a fair value for $-X$ (i.e. an expectation or prevision $P(-X)$) and introduces a *loading*, often in a multiplicative form, getting in this case a final price $\overline{P}(-X) = (1+k)P(-X)$, $k > 0$. In [6], the term Dutch premium principle identifies a "double loading" rule, which in our setting can be written as:

$$\rho_{DL} = (1+k)E[\min(-X, d_X)] + (1+c)E[(-X-d_X)_+] \tag{10}$$

with $k, c \geq 0$.

The idea in (10) is that the risk ensuing from $X$ is split between an insurer, which is liable until the threshold $d_X$, and a reinsurer liable for the residual risk, and both ask for their own loading to be payed by the insured. It is shown in [6] that requiring some reasonable properties reduces $\rho_{DL}$ to $\rho'_D$ with $d = 1$ in (4) and $d_X = E(-X)$, $k = 0$ in (10).

The last constraint, $k = 0$, was interpreted in [6] as impossibility of double loading without violating a condition (*no rip-off*) corresponding to b1) in Proposition 1.

What does Equation (5) tell us about this problem? If $\rho(X)$ is greater than the fair value $P(-X)$ it incorporates a loading on $X$. Then double loading is feasible while obtaining a final measure $\rho_c(X)$ (a premium) which is either coherent or convex, under the assumptions of Propositions 3 or 5, respectively. That is, under these assumptions $\rho_c(X)$ is guaranteed to keep adequate consistency properties and is a generalization of the Dutch risk measure. It is intuitively plausible that the condition for double loading, $\rho(X) > P(-X)$, should hold. The argument may be made more precise when $\rho$ is coherent. In fact, $\rho(X)$ is an upper prevision for $-X$, $\rho(X) = \overline{P}(-X)$. From Proposition 2 we know that $\rho(X) \geq P(-X)$, $\forall P \in \mathcal{M}$, where $\mathcal{M}$ is naturally interpreted as a set containing the "true" (although possibly unknown) prevision $P_0$ for $-X$. Thus typically $\rho(X) > P_0(-X)$.

Let us now consider the risk measure $\rho'_D(X)$ in (4), with $d \neq 1$, which was also employed in some papers, including [7]. It is known that $\rho'_D(X)$ satisfies the translation invariance property c1) in Proposition 1 if and only if $d = 1$ [6]. Therefore this measure does not meet the consistency requirement of convexity if $d \neq 1$. Further, its not following translation invariance prevents $\rho'_D(X)$ (and potential analogous generalizations of (5) with $-d\rho(X)$ replacing $-\rho(X)$) from meeting even the much weaker (and in a sense minimal, since it generalizes properties of capacities) consistency notion of centered 1-convexity considered in the next section. Therefore, this kind of generalization does not seem adequate for risk measurement.

Finally, we note that putting $d = 1$ in (4) reduces the second expectation to $E[(E(X) - X)_+]$, which is also a (mild generalization of a) *deviation* measure, following [12]. The correspondence is not necessarily true for the more general term $\overline{P}[(-\rho(X) - X)_+]$ in (5); for instance it does not hold when $\overline{P}$ is C-convex.

## 4 1-convex and shortfall-dependant imprecise previsions

In the realm of imprecise previsions, the shortfall-dependant measures $\rho_c$ obtained by equation (5) correspond to the lower previsions $\underline{P}_c$ defined by equation (6). Thus equation (6) displays a method for getting a more prudential lower prevision $\underline{P}_c$ from and by means of a previously assessed $\underline{P}_1$.

We might employ for instance (6) when $\underline{P}_1$ is someone else's prevision, considered not fully reliable by us.

The question we are concerned with in this section is: can Proposition 4 be generalized, meaning that asking $\underline{P}_1$, $\underline{P}_2$ to obey weaker consistency requirements than coherence (or convexity), $\underline{P}_c$ satisfies the same consistency conditions? We shall see that the answer is affirmative for a rather mild consistency notion, namely (centered) 1-convexity.

**Definition 4** *A map $\underline{P} : D \to \mathbb{R}$ is a 1-*convex *lower prevision on $D$ iff, $\forall X, Y \in D$*

$$\sup[(X - \underline{P}(X)) - (Y - \underline{P}(Y))] \geq 0 \qquad (11)$$

*A 1-convex lower prevision is* centered *if ($0 \in D$ and) $\underline{P}(0) = 0$.*

We chose the wording "1-convex", exhibiting some alikeness with "1-coherence" in [14](Appendix B), because Definition 4 is actually obtained by imposing $n = 1$ in Definition 2 (putting $s_0 = s_1 = 1$ instead of $s_0 = s_1 = k > 0$ is immaterial for the condition $\sup \underline{G} \geq 0$), i.e. it corresponds to checking convexity only when $n = 1$ (similarly, 1-coherence requires $n = 1$ in Definition 1, plus an extra condition not involved here).

There are other ways of expressing 1-convexity:

**Lemma 2** *Given $\underline{P} : D \to \mathbb{R}$,*
*a) Condition (11) is equivalent to:*

$$X \geq Y + c \Rightarrow \underline{P}(X) \geq \underline{P}(Y) + c, \forall X, Y \in D, \forall c \in \mathbb{R} \qquad (12)$$

*b) if $D$ is a linear space, condition (11) is equivalent to translation invariance plus monotonicity, i.e. axioms c1) and c2) in Proposition 1.*

**Proof.** We prove a) (a proof of b) was given in [4]). Suppose (12) holds. Applying it to $X - \sup(X - Y) \leq Y$ we get $\underline{P}(X) - \sup(X - Y) \leq \underline{P}(Y)$, from which (11) follows.

Conversely, let $X \geq Y + c$, hence $-c - \underline{P}(Y) + \underline{P}(X) \geq Y - \underline{P}(Y) - (X - \underline{P}(X))$. Using also (11), $-c - \underline{P}(Y) + \underline{P}(X) \geq \sup[(Y - \underline{P}(Y)) - (X - \underline{P}(X))] \geq 0$, which implies (12). □

Condition (12) is helpful in making a direct comparison between 1-convexity and coherence when $D$ is a convex cone. In fact, in this instance coherence of $\underline{P}$ is equivalent to its jointly satisfying conditions b2), b3) (cf. Proposition 1, b)) and (12) (see [14], p.76).

A remarkable consequence of Lemma 2 is that:

**Proposition 6** *Let $\underline{P}$ be a centered 1-convex lower prevision defined on the powerset $2^{\Omega}$ of a finite partition $\Omega$ of events. Then $\underline{P}$ is a capacity.*

**Proof.** Since $\underline{P}$ is centered, $\underline{P}(0) = 0$. By putting $X = 1$ and $Y = 0$ (0, 1 are the indicators of $\emptyset$, $\Omega$) in Definition 4, we get easily $\underline{P}(1) \leq 1$, whilst the reverse inequality is established by interchanging $X$ and $Y$. Hence $\underline{P}(1) = 1$. Monotonicity is implied by Lemma 2,a), with $c = 0$. □

Since the proof above is independent of the domain on which $\underline{P}$ is defined, any centered 1-convex $\underline{P}$ is normalized and monotone. It is however not necessarily lower or upper semicontinuous, thus being generally not a fuzzy measure when $\Omega$ is infinite.

Functionals defined from a linear space into the compact real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and for which translation invariance and monotonicity hold were termed *niveloids* in [4]. By Lemma 2,b) finite-valued niveloids are 1-convex previsions.

In order to generalize Proposition 4, we preliminarily check whether the results needed for its proof hold for 1-convex previsions too.

It is easy to realize that:

**Lemma 3** *If $\underline{P}_1$, $\underline{P}_2$ are 1-convex (or centered 1-convex) on $D$, then so is $\underline{P} = \lambda\underline{P}_1 + (1-\lambda)\underline{P}_2$, $\lambda \in [0,1]$ (cf. Proposition 1,d)).*

**Proof.** Condition (11) in Definition 4 is equivalent to $\underline{P}(X) - \underline{P}(Y) \leq \sup(X-Y)$ and holds for $\underline{P}_1$ and $\underline{P}_2$. Then, since $\underline{P}(X)-\underline{P}(Y) = \lambda(\underline{P}_1(X)-\underline{P}_1(Y)) + (1-\lambda)(\underline{P}_2(X)-\underline{P}_2(Y)) \leq \lambda\sup(X-Y) + (1-\lambda)\sup(X-Y) = \sup(X-Y)$, $\forall X,Y \in D$, condition (11) holds also for $\underline{P}$. □

The corresponding generalization of Proposition 1,a) is less immediate. We cannot simply apply an extension theorem for niveloids given in [4] to 1-convex lower previsions, because it does not guarantee that the extension is *finite*, as we need it to be. We proceed then by proving that there always exists a special extension, the *1-convex natural extension* (the notion resembles that of natural extension in [14] or of convex natural extension in [9]):

**Definition 5** *Given $\underline{P} : D \to \mathbb{R}$ and a linear space $\mathcal{L} \supset D$, define, for each $Z \in \mathcal{L}$,*
*$L(Z) = \{\alpha \in \mathbb{R} \mid Z - \alpha \geq X - \underline{P}(X)$, for some $X \in D\}$.*
*$\underline{E}(Z) = \sup_\alpha L(Z)$ is the* 1-convex natural extension *of $\underline{P}$ on $Z$ ($\underline{E}(Z) = -\infty$ if $L(Z) = \emptyset$).*

When $\underline{P}$ is 1-convex, it is easy to prove that the notion of 1-convex natural extension coincides with that of upper projection of a niveloid given in [4] and, therefore, $\underline{E}$ is a niveloid such that $\underline{E}(X) = \underline{P}(X), \forall X \in D$. We prove that $\underline{E}(Z) \in \mathbb{R}$ $\forall Z \in \mathcal{L}$, hence $\underline{E}$ is a 1-convex extension of $\underline{P}$ to $\mathcal{L}$.

**Proposition 7** *Given a 1-convex lower prevision $\underline{P} : D \to \mathbb{R}$, and a linear space $\mathcal{L} \supset D$, $\underline{E}(Z) \in \mathbb{R}$ $\forall Z \in \mathcal{L}$ and $\underline{E}$ is a 1-convex extension of $\underline{P}$.*

**Proof.** Let $Z \in \mathcal{L}$, $X \in D$ and $\overline{\alpha} = \inf Z - \sup X + \underline{P}(X)$. Hence, $Z - \overline{\alpha} = Z - \inf Z + \sup X - \underline{P}(X) \geq X - \underline{P}(X)$, which implies $\overline{\alpha} \in L(Z)$ (hence $L(Z)$ is non-empty) and $\underline{E}(Z) \geq \inf Z - \sup X + \underline{P}(X) > -\infty$. We show now that $\underline{E}(Z) < +\infty$. If $\alpha \in L(Z)$, there exists $X \in D$ such that $Z - \alpha \geq X - \underline{P}(X)$. Therefore, for any $Y \in D$, $\sup Z - \alpha \geq \sup X - \underline{P}(X) = \sup(X - \inf Y) - \underline{P}(X) + \inf Y \geq \sup(X-Y) - \underline{P}(X) + \inf Y \geq \underline{P}(X) - \underline{P}(Y) - \underline{P}(X) + \inf Y = \inf Y - \underline{P}(Y)$, where (11) is employed in the last inequality. It follows $\underline{E}(Z) \leq \sup Z + \underline{P}(Y) - \inf Y < +\infty$. Since $\underline{E}$ is a niveloid coinciding with $\underline{P}$ on $D$ [4] and $\underline{E}(Z) \in \mathbb{R}$ $\forall Z \in \mathcal{L}$, $\underline{E}$ is a 1-convex extension of $\underline{P}$ to $\mathcal{L}$. □

We can now prove the final result of the section.

**Proposition 8** *Let $\underline{P}_1$, $\underline{P}_2$ be two 1-convex lower previsions on $D_1$, $D_2 \supset \{Y : Y = \min(X+h,k), X \in D_1, h,k \in \mathbb{R}\}$ respectively.*
*Then $\underline{P}_c(X) = \underline{P}_1(X) + c\underline{P}_2[(X - \underline{P}_1(X))_-]$, $c \in [0,1]$ is a 1-convex lower prevision on $D_1$. If $\underline{P}_1$ and $\underline{P}_2$ are centered, then so is $\underline{P}_c$.*

**Proof.** We can follow the guidelines of the proof of Proposition 5. Observe first that we can always suppose that the relevant 1-convex lower previsions are defined and 1-convex on a linear space $\mathcal{L}$ by extending them to $\mathcal{L}$, if necessary. By Proposition 7, there exists a 1-convex extension to $\mathcal{L}$ of a 1-convex lower prevision. Thus $\underline{P}^*(X)$ in equation (7) may be defined on $\mathcal{L}$ with $\underline{P}_1$, $\underline{P}_2$ 1-convex on $\mathcal{L}$. Then $\underline{P}^*(X)$ is 1-convex by Lemma 2,b), because it satisfies c1) and c2) (a proof for this fact is already contained in the proof of Proposition 5). We note then that $\underline{P}_c(X)$ can be decomposed as in (9) when $\underline{P}_1$ and $\underline{P}^*$ are 1-convex. Hence $\underline{P}_c$ is 1-convex by Lemma 3. Finally, if $\underline{P}_1(0) = \underline{P}_2(0) = 0$ then also $\underline{P}^*(0) = \underline{P}_c(0) = 0$. □

## 5 Conclusions

Resorting to the theory of imprecise previsions, we have generalized a method, originally devised in an insurance pricing framework, for obtaining a second–choice uncertainty measure on the basis of the potential inadequacy of a formerly defined measure.

In particular, as shown in Section 3.1, it is possible (and convenient) to assess independently the initial measure and the measure of its shortfall, which jointly determine the final measure through equation (5).

As a further advantage, the resulting measure conforms to the well established consistency notions of coherence or (centered) convexity, provided that the other measures in the procedure comply with the same (or more stringent) requirements.

As a practical application in the framework of insurance pricing, this approach provides a formal support to the policy of double loading, while ensuring the desirable properties guaranteed by the various consistency notions.

The choice of the consistency criterion to be employed may depend on many factors. Undoubtedly coherence seems preferable [1, 14], but arguments in favour of convexity or centered convexity were also brought forth [5, 9]. 1–convexity is probably too weak for risk measurement, but could be useful for other kinds of applications in the realm of imprecise previsions. It is anyway interesting to notice that the method summarized by (5) can be applied as far as to consider 1-convexity. The generalization of (5) in Proposition 8 seems the largest operationally relevant: 1-convexity is a really minimal consistency requirement, as can be seen from the displayed comparisons with the concepts of capacity and niveloid.

Further extensions of this work should therefore address different questions. An appealing and largely unexplored area is that of investigating shortfall-based conditional risk measures (and previsions). Here the set $D$ should be made of conditional random variables like $X|B$, where $B$ is a non-impossible event and the conditioning events for the variables in $D$ are generally different. Notions of coherence and C-convexity with related fundamental properties are available in such a framework [10, 11, 14], and equation (2) easily generalises to $\rho(X|B) = \overline{P}(-X|B) = -\underline{P}(X|B)$. The point is how should equation (5) be generalised to guarantee some properties that are similar to those of Propositions 4 and 5. An immediate difficulty is that the proof of these propositions relies on Proposition 1 d), which is known to admit no analogue in the conditional environment.

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